

GLOBAL SOLVABILITY OF THE ROTATING NAVIER-STOKES EQUATIONS WITH FRACTIONAL LAPLACIAN IN A PERIODIC DOMAIN

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ABSTRACT. We consider existence of global solutions to equations for three-dimensional rotating fluids in a periodic frame provided by a sufficiently large Coriolis force. The Coriolis force appears in almost all of the models of meteorology and geophysics dealing with large-scale phenomena. In the spatially decaying case, Koh, Lee and Takada (2014) showed existence for the large times of solutions of the rotating Euler equations provided by the large Coriolis force. In this case the resonant equation does not appear anymore. In the periodic case, however, the resonant equation appears, and thus the main subject in this case is to show existence of global solutions to the resonant equation. Research in this direction was initiated by Babin, Mahalov and Nicolaenko (1999) who treated the rotating Navier-Stokes equations on general periodic domains. On the other hand, Golse, Mahalov and Nicolaenko (2008) considered bursting dynamics of the resonant equation in the case of a cylinder with no viscosity. Thus we may not expect to show global existence of solutions to the resonant equation without viscosity in the periodic case. In this paper we show existence of global solutions for fractional Laplacian case (with its power strictly less than the usual Laplacian) in the periodic domain with the same period in each direction. The main ingredient is an improved estimate on resonant three-wave interactions, which is based on a combinatorial argument.

1. INTRODUCTION

We consider the rotating three-dimensional Navier-Stokes equations with the fractional Laplacian:

$$(1.1) \quad \begin{aligned} \partial_t u + (u \cdot \nabla)u + \Omega e_3 \times u + \nu(-\Delta)^\alpha u &= -\nabla p \quad \text{in } \mathbb{T}^3 := [0, 2\pi]^3, \\ \nabla \cdot u &= 0 \quad \text{and} \quad u|_{t=0} = u_0, \end{aligned}$$

where $u = u(t) = (u^1(t, x), u^2(t, x), u^3(t, x))$ is the unknown velocity vector field and $p = p(t, x)$ is the unknown scalar pressure at the point $x = (x_1, x_2, x_3) \in [0, 2\pi]^3$ in space and time $t > 0$ while $u_0 = u_0(x)$ is the given initial velocity field. Here $\Omega \in \mathbb{R}$ is the Coriolis parameter, which is twice the angular velocity of the rotation around the vertical unit vector $e_3 = (0, 0, 1)$, and $\nu > 0$ is the kinematic viscosity coefficient. By \times we denote the exterior product, and hence, the Coriolis term is

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represented by $e_3 \times u = Ju$ with the corresponding skew-symmetric 3×3 matrix J , namely,

$$J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Coriolis force plays a significant role in the large scale flows considered in meteorology and geophysics. In 1868 Kelvin observed that a sphere moving along the axis of uniformly rotating water takes with it a column of liquid as if this were a rigid mass (see [9] for references). After that, Taylor [25] and Proudman [24] did important contributions. Mathematically it was investigated by Poincaré [24], more recently, by Babin, Mahalov and Nicolaenko [2, 3] using the fully Navier-Stokes equations in a periodic domain.

Throughout this paper we essentially use the spatial Fourier transform denoted by \mathcal{F} or $\widehat{\cdot}$:

$$u(x) = \sum_{n \in \mathbb{Z}^3} \widehat{u}(n) e^{in \cdot x} \quad \text{with} \quad (\mathcal{F}u)(n) = \widehat{u}(n) := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} u(x) e^{-in \cdot x} dx.$$

Let us define the inhomogeneous Sobolev spaces H^s as follows:

$$H^s(\mathbb{T}^d) := \left\{ u = \sum_{n \in \mathbb{Z}^d} \widehat{u}(n) e^{in \cdot x} : \left(\sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^s |\widehat{u}(n)|^2 \right)^{1/2} =: \|u\|_{H^s} < \infty \right\}.$$

The homogeneous version \dot{H}^s can be defined as

$$\dot{H}^s(\mathbb{T}^d) := \left\{ u = \sum_{n \in \mathbb{Z}^d} \widehat{u}(n) e^{in \cdot x} : \left(\sum_{n \in \mathbb{Z}^d} |n|^{2s} |\widehat{u}(n)|^2 \right)^{1/2} =: \|u\|_{\dot{H}^s} < \infty \right\}.$$

We will assume that all the vector fields in this paper are mean-zero. This assumption is valid from the following observation: Let $f(t) \in \mathbb{R}^3$ be an average of the solution to (1.1) at t , that is, the solution to the following ODE:

$$f'(t) + \Omega J f(t) = 0, \quad f(0) = \text{average of } u_0 \text{ in } x.$$

Then the following invertible transforms

$$u(t, x) \mapsto u \left(t, x + \int_0^t f(s) ds \right) - f(t) \quad \text{and} \quad p(t, x) \mapsto p \left(t, x + \int_0^t f(s) ds \right)$$

preserve the equation (1.1), and the new velocity field has zero mean for all time. We therefore do not distinguish homogeneous and inhomogeneous Sobolev spaces.

Let us recall the result of Babin et al. as the starting point of our work:

Theorem 1.1 ([3]). *Let $s > 1/2$ and \mathbf{T}^3 be a torus with arbitrary period (distinguish from \mathbb{T}^3). Let $v_0 \in H^s(\mathbf{T}^3)$ be a divergence-free vector field. Then there exists a positive Ω_0 depending on $\|v_0\|_{H^s}$ and the period of torus such that for all $|\Omega| \geq \Omega_0$, there is a unique global solution*

$$u(t) \in C([0, \infty) : H^s(\mathbf{T}^3)) \cap L^2(0, \infty : H^{s+1}(\mathbf{T}^3))$$

to the equation (1.1) with $\alpha = 1$.

As shown in [2, 3], it turns out that the estimates on the obtained global solutions depend crucially on the period of the torus. For instance, the global a priori bound is independent of the viscosity coefficient $\nu > 0$ for generic periods ([2]), whereas exponential-in- ν^{-1} dependence may occur in the “worst case” ([3]). In this paper, we will focus on the special torus $\mathbb{T}^3 = [0, 2\pi]^3$, which is among the “worst case” as we will see in Section 4.4 below. We remark that the above result was extended to the critical case $s = 1/2$ in [7, Theorem 6.2]; see also [7, Theorem 5.7] for an analogous result on \mathbb{R}^3 .

We next recall previous results in the inviscid case. In the spatially decaying setting, combining the Strichartz estimates with Beale-Kato-Majda’s blow-up criterion, Koh, Lee and Takada [20] showed long time existence of solutions to the Euler equations provided by large Coriolis parameter. The periodic case may be more difficult due to the appearance of the resonant equation. In [2], Babin et al. initially considered long time solvability of the rotating Euler equations (see also [22] in a cylinder case). However they set specific periodic domains (specific aspect ratios) and eliminate “nontrivial resonant part” which is essentially related to the Rossby wave in physics (see [19, 27] for example). For domains with other periods we need to deal with “nontrivial resonant part”, and it has been an open problem. On the other hand, in a cylinder case, Golse, Mahalov and Nicolaenko [16] considered bursting dynamics of the inviscid resonant equation. Thus we may not expect to show existence of inviscid smooth global flow in general periodic cases. Nevertheless, by a refined estimate on “nontrivial resonant part” based on elementary number theory (Lemma 5.1 below), we can progress a less viscosity effect case (fractional Laplacian case) in the periodic domain $\mathbb{T}^3 = [0, 2\pi]^3$. A fractional Laplacian (superviscosity), though it has little physical meaning, has been employed in many numerical works instead of the usual viscosity; see [26] for example.

First we state the following local existence theorem, which is obtained by a standard argument:

Theorem 1.2. *Let $\nu > 0$, $\alpha \in (\frac{3}{4}, 1]$, $s \geq 1$. Then, (1.1) is locally well-posed in $H^s(\mathbb{T}^3)$: For any $u_0 \in H^s$ with $\operatorname{div} u_0 = 0$, there exist $T_L > 0$ and a unique solution $u \in C([0, T_L]; H^s) \cap C((0, T_L]; H^\infty)$ of (1.1) such that*

$$(1.2) \quad T_L \geq c(\alpha) \nu^{\frac{3}{4\alpha-3}} \|u_0\|_{H^1}^{-\frac{4\alpha}{4\alpha-3}},$$

$$(1.3) \quad \sup_{0 \leq t \leq T_L} \|u(t)\|_{H^s} \leq C(s) \|u_0\|_{H^s}.$$

Moreover, there exists $\eta = \eta(s, \alpha) > 0$ such that if $\|u_0\|_{H^1} \leq \eta(1, \alpha)\nu$, then the solution u is global. If $\|u_0\|_{H^1} \leq \eta(s, \alpha)\nu$, then it holds that

$$\|u(t)\|_{H^s} \leq e^{-\frac{1}{2}\nu t} \|u_0\|_{H^s}, \quad t \geq 0.$$

We give its proof in the next section. Since we consider the subcritical problem with respect to the scaling (see Section 7.2), time of local existence is bounded from below in terms of the size of initial data. Note also that the result is uniform in the Coriolis parameter Ω .

We now state the main theorem. For $\alpha \in (\frac{3}{4}, 1]$, $C(\alpha)$ denotes any positive constant depending on α with $C(\alpha) \rightarrow \infty$ as $\alpha \downarrow \frac{3}{4}$. Our main result is as follows.

Theorem 1.3. *Let $\nu > 0$ and $\alpha \in (\frac{3}{4}, 1]$. For any $E > 0$, there exists Ω_0 depending on α , ν and E such that for any $\Omega \in \mathbb{R}$ with $|\Omega| \geq \Omega_0$ and any real-valued*

divergence-free mean-zero initial vector field $u_0 \in H^1(\mathbb{T}^3)$ with $\|u_0\|_{H^1} \leq E$, there exists a unique global smooth solution $u(t)$ of (1.1). Moreover, Ω_0 can be taken as

$$\Omega_0 = (\nu + E) \exp \exp \left(C(\alpha) \left(1 + \frac{E}{\nu}\right)^{C(\alpha)} \right).$$

Remark 1.4. For $a = (a_1, a_2, a_3) \in (0, \infty)^3$, consider the torus

$$\mathbf{T}_a^3 := [0, 2\pi a_1) \times [0, 2\pi a_2) \times [0, 2\pi a_3).$$

We say \mathbf{T}_a^3 regular if $a_1 = a_2 = a_3$ or rational if $a_2^2/a_1^2, a_3^2/a_1^2 \in \mathbb{Q}$. Although we will mainly focus on the case of $\mathbb{T}^3 = [0, 2\pi)^3$, our result can be extended to the case of any *rational* periodic domains with a slight modification. See Remark 5.5 below.

We also remark that for periodic domains \mathbf{T}_a^3 with $a_2/a_1, a_3/a_1 \in \mathbb{Q}$, global regularity under fast rotation follows immediately from the above theorem and a scaling argument. In fact, by a suitable scaling transformation, any solution (u, p) to (1.1) on \mathbf{T}_a^3 with $a_2/a_1, a_3/a_1 \in \mathbb{Q}$ can be transformed into a solution on \mathbb{T}^3 with rescaled initial data and rotation speed.

Remark 1.5. We will obtain a global H^1 bound on the solution to (1.1) which is polynomial in ν^{-1} rather than exponential. Even for the usual Navier-Stokes equations ($\alpha = 1$) this improves the previous result of Babin et al. [3] in the case of rational periodic domains. See Section 4.4 for related discussion.

Remark 1.6. Let us define the function space $\mathcal{F}^{-1}\ell_1$ as follows:

$$\mathcal{F}^{-1}\ell_1(\mathbb{T}^d) := \left\{ u = \sum_{n \in \mathbb{Z}^d} \hat{u}(n) e^{in \cdot x} : \sum_{n \in \mathbb{Z}^d} |\hat{u}(n)| =: \|u\|_{\mathcal{F}^{-1}\ell_1} < \infty \right\}.$$

It is well known that $\mathcal{F}^{-1}\ell_1$ is continuously embedded in the space of continuous functions (in the nonperiodic case it is embedded in BUC , the space of bounded uniformly continuous functions). In the spatially almost periodic case, $\mathcal{F}^{-1}\ell_1$ framework seems to be one of the most suitable (see [10, 11, 12, 13, 14, 15, 28] for example). On the other hand, to control the nontrivial resonant part, the energy method is one of the most powerful tools. Note that, up to now, we can use the energy method only in the periodic case, thus, controlling the nontrivial resonant part in the spatially almost periodic case is open. See also Section 7.2.

In the rest of this section, we outline the proof of the main theorem (Theorem 1.3). We basically follow the previous argument in [3, 7].

The Poincaré propagator $\mathcal{L}(\Omega t) = e^{-\Omega t \mathbb{P} J \mathbb{P}}$ is defined as the unitary group associated with the linear problem

$$\partial_t \Phi + \Omega \mathbb{P}(e^3 \times \Phi) = 0, \quad \Phi|_{t=0} = \Phi_0 \quad \text{with} \quad \operatorname{div} \Phi_0 = 0,$$

where \mathbb{P} denotes the Helmholtz-Leray projection onto divergence-free vector fields; \mathbb{P} acts as multiplication by the matrix $\hat{\mathbb{P}}(n)$ in the Fourier space:

$$\hat{\mathbb{P}}(n) = \operatorname{Id} - \left(\frac{n_i n_j}{|n|^2} \right)_{1 \leq i, j \leq 3} = \frac{1}{|n|^2} \begin{pmatrix} n_2^2 + n_3^2 & -n_1 n_2 & -n_1 n_3 \\ -n_2 n_1 & n_1^2 + n_3^2 & -n_2 n_3 \\ -n_3 n_1 & -n_3 n_2 & n_1^2 + n_2^2 \end{pmatrix}.$$

We see that the matrix

$$\widehat{\mathbb{P}}(n)J\widehat{\mathbb{P}}(n) = \frac{n_3}{|n|^2} \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

has eigenvalues $\pm i \frac{n_3}{|n|}$, 0, and for each $n \in \mathbb{Z}^3 \setminus \{0\}$, the vectors $e^\pm(n) \in \mathbb{C}^3$ defined by

$$e^\pm(n) = \begin{cases} \frac{1}{\sqrt{2}|n||n^h|} \begin{pmatrix} n_1 n_3 \pm i n_2 |n| \\ n_2 n_3 \mp i n_1 |n| \\ -|n^h|^2 \end{pmatrix} & \text{if } n^h := (n_1, n_2) \neq 0, \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \operatorname{sgn}(n_3) \\ 0 \end{pmatrix} & \text{if } n^h = 0 \end{cases}$$

are eigenvectors corresponding to $\pm i \frac{n_3}{|n|}$ and form an orthonormal basis of

$$\{\widehat{a} \in \mathbb{C}^3 \mid n \cdot \widehat{a} = 0\} = \operatorname{Ran} \widehat{\mathbb{P}}(n).$$

Therefore, the Poincaré propagator $\mathcal{L}(\Omega t)$ acts on a divergence-free and mean-free vector field $a(x) = \sum_{n \neq 0} \widehat{a}(n) e^{in \cdot x}$ as

$$[\mathcal{L}(\Omega t)a](x) = \sum_{n \neq 0} \sum_{\sigma \in \{\pm\}} e^{-\sigma i \Omega t \frac{n_3}{|n|}} \widehat{a}^\sigma(n) e^{in \cdot x}, \quad \widehat{a}^\sigma(n) := (\widehat{a}(n) | e^\sigma(n)) e^\sigma(n),$$

where $(\cdot | \cdot)$ denotes the inner product of \mathbb{C}^3 . (Note that $a \cdot b = \sum_j a_j b_j$ whereas $(a|b) = \sum_j a_j b_j^*$, where $*$ stands for the complex conjugate.) It is also easy to see that

$$(1.4) \quad n \times e^\pm(n) = \pm i |n| e^\pm(n), \quad n \in \mathbb{Z}^3 \setminus \{0\}.$$

Using this, we obtain another representation of $\mathcal{L}(\Omega t)$:

$$[\mathcal{L}(\Omega t)a](x) = \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} \left[\cos\left(\frac{\Omega t n_3}{|n|}\right) \widehat{a}(n) - \sin\left(\frac{\Omega t n_3}{|n|}\right) \frac{n}{|n|} \times \widehat{a}(n) \right] e^{in \cdot x}.$$

In fact, for divergence-free and mean-zero a ,

$$\begin{aligned} & \cos\left(\frac{\Omega t n_3}{|n|}\right) \widehat{a}(n) - \sin\left(\frac{\Omega t n_3}{|n|}\right) \frac{n}{|n|} \times \widehat{a}(n) \\ &= \sum_{\sigma \in \{\pm\}} \frac{1}{2} e^{-\sigma i \Omega t \frac{n_3}{|n|}} \left(\widehat{a}(n) - \sigma i \frac{n}{|n|} \times \widehat{a}(n) \right) \\ &= \sum_{\sigma \in \{\pm\}} \frac{1}{2} e^{-\sigma i \Omega t \frac{n_3}{|n|}} \sum_{\sigma' \in \{\pm\}} (\widehat{a}(n) | e^{\sigma'}(n)) \left(e^{\sigma'}(n) - \sigma i \frac{n}{|n|} \times e^{\sigma'}(n) \right) \\ &= \sum_{\sigma \in \{\pm\}} e^{-\sigma i \Omega t \frac{n_3}{|n|}} \sum_{\sigma' \in \{\pm\}} (\widehat{a}(n) | e^{\sigma'}(n)) \frac{e^{\sigma'}(n) + \sigma \sigma' e^{\sigma'}(n)}{2} \\ &= \sum_{\sigma \in \{\pm\}} e^{-\sigma i \Omega t \frac{n_3}{|n|}} \widehat{a}^\sigma(n). \end{aligned}$$

Next, we set $v(t) := \mathcal{L}(-\Omega t)u(t)$. If $u(t)$ solves (1.1), then v (formally) solves

$$(1.5) \quad \begin{cases} \partial_t v + \nu(-\Delta)^\alpha v + B(\Omega t; v(t), v(t)) = 0, & t > 0, \quad x \in \mathbb{T}^3, \\ v|_{t=0} = u_0 \quad \text{with} \quad \operatorname{div} u_0 = 0, \end{cases}$$

where

$$B(\Omega t; a, b) := \mathcal{L}(-\Omega t) \mathbb{P}(\mathcal{L}(\Omega t) a \cdot \nabla) \mathcal{L}(\Omega t) b,$$

so that

$$\begin{aligned} & [\mathcal{F}B(\Omega t; a, b)](n) \\ &= i \sum_{\substack{\sigma=(\sigma_1, \sigma_2, \sigma_3) \\ \in \{\pm\}^3}} \sum_{\substack{k, m \neq 0 \\ n=k+m}} e^{-i\Omega t \omega_{nkm}^\sigma} (\widehat{a}^{\sigma_1}(k) \cdot m) (\widehat{b}^{\sigma_2}(m) | e^{\sigma_3}(n)) e^{\sigma_3}(n), \\ & \omega_{nkm}^\sigma := \sigma_1 \frac{k_3}{|k|} + \sigma_2 \frac{m_3}{|m|} - \sigma_3 \frac{n_3}{|n|}. \end{aligned}$$

Now we decompose $B(\Omega t; a, b)$ into the resonant and the non-resonant parts as

$$(1.6) \quad B(\Omega t; a, b) = B_R(a, b) + B_{NR}(\Omega t; a, b),$$

where

$$[\mathcal{F}B_R(a, b)](n) := i \sum_{\sigma \in \{\pm\}^3} \sum_{\substack{n=k+m \\ \omega_{nkm}^\sigma=0}} (\widehat{a}^{\sigma_1}(k) \cdot m) (\widehat{b}^{\sigma_2}(m) | e^{\sigma_3}(n)) e^{\sigma_3}(n),$$

so that

$$\begin{aligned} & [\mathcal{F}B_{NR}(\Omega t; a, b)](n) \\ &= i \sum_{\sigma \in \{\pm\}^3} \sum_{\substack{n=k+m \\ \omega_{nkm}^\sigma \neq 0}} e^{-i\Omega t \omega_{nkm}^\sigma} (\widehat{a}^{\sigma_1}(k) \cdot m) (\widehat{b}^{\sigma_2}(m) | e^{\sigma_3}(n)) e^{\sigma_3}(n). \end{aligned}$$

It is expected (and actually proved in Section 6) that only the resonant part contributes in the limit $|\Omega| \rightarrow \infty$. Therefore, we need to consider the following limit equation (*resonant equation*):

$$(1.7) \quad \begin{cases} \partial_t U + \nu(-\Delta)^\alpha U + B_R(U(t), U(t)) = 0, & t > 0, \quad x \in \mathbb{T}^3, \\ U|_{t=0} = u_0 \quad \text{with} \quad \operatorname{div} u_0 = 0. \end{cases}$$

We remark that similar local existence results to Theorem 1.2 for the equation (1.5) and for the limit equation (1.7) can be obtained with the identical proof.

The main task is to show existence of global regular solutions to the resonant equation (1.7). More precisely we will show the following:

Proposition 1.7. *Let $\nu > 0$, $\alpha \in (\frac{3}{4}, 1]$ and $u_0 \in H^1(\mathbb{T}^3)$ be any real-valued, divergence-free, mean-zero initial vector field. Then, there exists a unique global solution $U \in C([0, \infty); H^1) \cap L^2((0, \infty); H^{1+\alpha}) \cap C((0, \infty); H^\infty)$ to (1.7) satisfying (4.7) and (4.8) below with $E := \|u_0\|_{H^1}$, $0 < \varepsilon < 2\alpha - \frac{3}{2}$. In particular, for any $s \geq 1$, the H^s norm of $U(t)$ decays exponentially for large time.*

To prove the above proposition, we make further decomposition of B_R into the 2D part and the non-trivial resonance part. For a 3D-3C (three-dimensional three-component) vector field $a = (a_1, a_2, a_3)^T : \mathbb{T}^3 \rightarrow \mathbb{R}^3$, we define

- 2D-3C vector field \bar{a} by $\bar{a}(x^h) := \frac{1}{2\pi} \int_0^{2\pi} a(x) dx_3$,
or $\bar{a}(x^h) = \sum_{n_3=0} \hat{a}(n) e^{in \cdot x}$,
- 3D-3C vector field a_{osc} by $a_{\text{osc}}(x) := a(x) - \bar{a}(x^h)$,
or $a_{\text{osc}}(x) = \sum_{n_3 \neq 0} \hat{a}(n) e^{in \cdot x}$,
- 3D-2C vector field a^h by $a^h(x) := (a_1(x), a_2(x))^T$.

It is easily verified that for any divergence-free and mean-zero vector fields a, b ,

$$\begin{aligned} \overline{B_R(\bar{a}, b_{\text{osc}})} &= \overline{B_R(a_{\text{osc}}, \bar{b})} = B_R(\bar{a}, \bar{b})_{\text{osc}} = 0, \\ \overline{B_R(\bar{a}, \bar{b})} &= B_R(\bar{a}, \bar{b}) = \begin{pmatrix} \mathbb{P}_h(\bar{a}^h \cdot \nabla^h) \bar{b}^h \\ (\bar{a}^h \cdot \nabla^h) \bar{b}_3 \end{pmatrix}, \end{aligned}$$

where \mathbb{P}_h is the 2D Helmholtz projection, and $\nabla^h = (\partial_{x_1}, \partial_{x_2})^T$. Note that $\text{div}_h \bar{u}_0^h := \nabla^h \cdot \bar{u}_0^h = 0$ if $\text{div } u_0 = 0$. Moreover, we see (Lemma 3.1 below) that

$$\overline{B_R(a_{\text{osc}}, a_{\text{osc}})} = 0.$$

These properties imply that $\overline{B_R(U, U)} = B_R(\bar{U}, \bar{U})$. Consequently, the limit equation (1.7) can be decomposed into the following three equations:

$$(1.8) \quad \begin{cases} \partial_t \bar{U}^h + \nu(-\Delta_h)^\alpha \bar{U}^h + \mathbb{P}_h(\bar{U}^h \cdot \nabla^h) \bar{U}^h = 0, & t > 0, \quad x \in \mathbb{T}^2, \\ \bar{U}^h|_{t=0} = \bar{u}_0^h \quad \text{with } \text{div}_h \bar{u}_0^h = 0, \end{cases}$$

$$(1.9) \quad \begin{cases} \partial_t \bar{U}_3 + \nu(-\Delta_h)^\alpha \bar{U}_3 + (\bar{U}^h \cdot \nabla^h) \bar{U}_3 = 0, & t > 0, \quad x \in \mathbb{T}^2, \\ \bar{U}_3|_{t=0} = \bar{u}_{0,3}, \end{cases}$$

$$(1.10) \quad \begin{cases} \partial_t U_{\text{osc}} + \nu(-\Delta)^\alpha U_{\text{osc}} \\ \quad + B_R(\bar{U}, U_{\text{osc}}) + B_R(U_{\text{osc}}, \bar{U}) + B_R(U_{\text{osc}}, U_{\text{osc}}) = 0, & t > 0, \quad x \in \mathbb{T}^3, \\ U_{\text{osc}}|_{t=0} = u_{0,\text{osc}} \quad \text{with } \text{div } u_{0,\text{osc}} = 0, \end{cases}$$

where $(-\Delta_h)^\alpha f := \mathcal{F}^{-1}[(n_1^2 + n_2^2)^\alpha \hat{f}]$.

The H^1 energy estimate for the 2D part $\bar{U}(t)$ can be obtained straightforwardly (see Section 4):

$$(1.11) \quad \|\bar{U}(t)\|_{H^1}^2 + \nu \int_0^t \|\bar{U}(\tau)\|_{H^{1+\alpha}}^2 d\tau \leq C(\|\bar{U}(0)\|_{H^1}) < \infty.$$

The key is to control the following norm globally in time (see Section 4):

$$(1.12) \quad \|U_{\text{osc}}(t)\|_{H^1}^2 + \nu \int_0^t \|U_{\text{osc}}(\tau)\|_{H^{1+\alpha}}^2 d\tau.$$

In order to control the above quantity in the weak viscosity case ($\alpha < 1$), we essentially use a new estimate on non-trivial resonant three-wave interactions (Lemma 4.1 below). We prove this estimate in Section 5 by using some tools from elementary number theory, which is the crucial idea in this paper. However, this kind of argument is only available for the case of *regular* (or *rational*) torus. As a result,

our main theorem is also restricted to that case. Once we get the above energy estimates, we will be able to deduce Proposition 1.7. See Section 4 for details.

To prove the main theorem, we first decompose the time interval $[0, \infty)$ into three parts:

- $[0, T_L)$: dominant linear part,
- $[T_L, T_C)$: dominant Coriolis force part,
- $[T_C, \infty)$: exponentially decaying part.

What we need for the first time interval $[0, T_L)$ is nothing more than the local existence result. By the global existence for small initial data, we can also control the solution to (1.1) in $[T_C, \infty)$ as long as it becomes sufficiently small at $t = T_C$. On the other hand, by (4.8) the solution to the resonant equation eventually becomes arbitrarily small. Thus, our main task is to ensure under the large Coriolis parameter assumption that solutions to the original equation and the resonant equation that coincide at $t = T_L$ stay very close to each other until $t = T_C$, which also means that we can obtain the existence theorem in $[T_L, T_C)$. To this end, it suffices to control the non-resonant part in (1.6). More precisely, our task is to estimate the difference $w(t) := v(t) - U(t)$, which satisfies

$$\begin{cases} \partial_t w + \nu(-\Delta)^\alpha w + B_R(w, v) + B_R(U, w) + B_{NR}(\Omega t; v, v) = 0, & t > T_L, \\ w|_{t=T_L} = 0. \end{cases}$$

Let \tilde{E} be the global upper bound of $U(t)$ in H^1 obtained in (4.7), and let \tilde{T}_L be a local existence time of the H^1 solution to (1.5) of size $2\tilde{E}$. The following lemma enables us to control the non-resonant part B_{NR} :

Lemma 1.8. *For any $\delta > 0$, there exists $\Omega_0 = \Omega_0(\delta, \alpha, \nu, E) > 0$ such that the following holds for $|\Omega| \geq \Omega_0$. Let K be the positive integer satisfying*

$$T_L + (K - 1)\tilde{T}_L < T_C \leq T_L + K\tilde{T}_L,$$

and let $k \in \{0, 1, \dots, K - 1\}$. Assume that $\|v(t)\|_{H^1} \leq 2\tilde{E}$ for $T_L \leq t \leq T_L + k\tilde{T}_L$. Then, we have

$$\begin{aligned} & \left| 2 \int_{T_L}^t \langle B_{NR}(\Omega t'; v(t'), v(t')), w(t') \rangle_{H^1} dt' \right| \\ & \leq \delta + \frac{1}{2} \left(\|w(t)\|_{H^1}^2 + \nu \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' \right) \end{aligned}$$

for $T_L \leq t \leq T_L + (k + 1)\tilde{T}_L$.

Roughly saying, we can control the contribution from the non-resonant forcing term by an arbitrarily small constant δ . By the above lemma, which will be restated as Lemma 6.1 and proved in Section 6, we can prove the main theorem (Theorem 1.3). For the precise argument, see Section 6.

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2. PROOF OF LOCAL WELL-POSEDNESS

In this section, we shall establish local well-posedness for (1.1), Theorem 1.2. Let us consider the corresponding integral equation:

(2.1)

$$u(t) = S(t)u_0 - \int_0^t S(t-t')\mathbb{P}(u(t') \cdot \nabla)u(t') dt', \quad S(t) := e^{-\nu t(-\Delta)^\alpha} \mathcal{L}(\Omega t).$$

Lemma 2.1. *Let $\nu, \alpha > 0$, $\Omega \in \mathbb{R}$, $s \in \mathbb{R}$ and $\theta \geq 0$. Then, for divergence-free and mean-zero vector fields a , it holds that*

$$\|S(t)a\|_{H^{s+\theta}} \lesssim (\nu t)^{-\frac{\theta}{2\alpha}} \|a\|_{H^s}, \quad t > 0.$$

The implicit constant depends only on $\frac{\theta}{2\alpha}$ and is independent of ν, t, s, Ω .

Remark 2.2. In the following argument, we apply this lemma only with $0 \leq \theta < 2\alpha$. Then, we can take the implicit constant independent of α and θ .

Proof. We see that

$$\begin{aligned} \|S(t)a\|_{H^{s+\theta}} &= \left\| |n|^\theta e^{-\nu t|n|^{2\alpha}} \cdot |n|^s [\mathcal{FL}(\Omega t)a](n) \right\|_{\ell^2(\mathbb{Z}^3)} \\ &\lesssim \left\| |n|^\theta e^{-\nu t|n|^{2\alpha}} \right\|_{\ell^\infty(\mathbb{Z}^3)} \|a\|_{H^s} \leq \left[\sup_{r>0} r^{\frac{\theta}{2\alpha}} e^{-r} \right] (\nu t)^{-\frac{\theta}{2\alpha}} \|a\|_{H^s}. \quad \square \end{aligned}$$

We will apply fixed point argument in the norm

$$\|f\|_{X_T^s} := \sup_{0 < t \leq T} \left(\|f(t)\|_{H^s} + (\nu t)^{\frac{\beta}{2\alpha}} \|f(t)\|_{H^{s+\beta}} \right)$$

for suitable $\beta > 0$ to be chosen later. By Lemma 2.1, we have

$$\|S(t)u_0\|_{X_T^s} \lesssim \|u_0\|_{H^s}, \quad T > 0.$$

For the Duhamel term, we want to estimate by using Lemma 2.1 as

$$\begin{aligned} &\left\| \int_0^t S(t-t')\mathbb{P}(u(t') \cdot \nabla)v(t') dt' \right\|_{H^s} \\ &\lesssim \int_0^t [\nu(t-t')]^{-\frac{1}{2\alpha}} \|u \otimes v(t')\|_{H^s} dt' \\ &\lesssim_{s,\beta} \int_0^t [\nu(t-t')]^{-\frac{1}{2\alpha}} \|u(t')\|_{H^{s+\frac{\beta}{2}}} \|v(t')\|_{H^{s+\frac{\beta}{2}}} dt' \\ &\lesssim \int_0^t [\nu(t-t')]^{-\frac{1}{2\alpha}} (\nu t')^{-\frac{\beta}{2\alpha}} dt' \|u\|_{X_T^s} \|v\|_{X_T^s} \\ &\lesssim_{\alpha,\beta} \nu^{-1} (\nu T)^{1-\frac{1+\beta}{2\alpha}} \|u\|_{X_T^s} \|v\|_{X_T^s} \end{aligned}$$

for $0 < t \leq T$, where we have used the divergence-free condition for u so that $(u \cdot \nabla)v = \nabla(u \otimes v)$, and we have also used the Sobolev estimate

$$\|fg\|_{H^s} \lesssim_{s,\beta} \|f\|_{H^{s+\frac{\beta}{2}}} \|g\|_{H^{s+\frac{\beta}{2}}},$$

which holds if $\beta > 0$ and $s+\beta \geq \frac{3}{2}$ (applying Lemma 7.1 below with $s_1 = s_2 = s + \frac{\beta}{2}$ and $s_3 = -s$). Hence, we need to assume

$$(2.2) \quad \beta > 0, \quad s + \beta \geq \frac{3}{2}, \quad 1 + \beta < 2\alpha$$

for the above argument. Similarly,

$$\begin{aligned}
& (\nu t)^{\frac{\beta}{2\alpha}} \left\| \int_0^t S(t-t') \mathbb{P}(u(t') \cdot \nabla) v(t') dt' \right\|_{H^{s+\beta}} \\
& \lesssim (\nu t)^{\frac{\beta}{2\alpha}} \int_0^t [\nu(t-t')]^{-\frac{1+\beta}{2\alpha}} \|u \otimes v(t')\|_{H^s} dt' \\
& \lesssim_{s,\beta} (\nu t)^{\frac{\beta}{2\alpha}} \int_0^t [\nu(t-t')]^{-\frac{1+\beta}{2\alpha}} (\nu t')^{-\frac{\beta}{2\alpha}} dt' \|u\|_{X_T^s} \|v\|_{X_T^s} \\
& \lesssim_{\alpha,\beta} \nu^{-1} (\nu T)^{1-\frac{1+\beta}{2\alpha}} \|u\|_{X_T^s} \|v\|_{X_T^s}
\end{aligned}$$

for $0 < t \leq T$, whenever (2.2) holds.

If $\alpha > \frac{3}{4}$, we can take $\beta = \max\{\frac{3}{2} - s, 0 +\}$ to fulfill (2.2). Then, the contraction mapping argument can be applied if we take $T > 0$ so that

$$\nu^{-1} (\nu T)^{1-\frac{1+\beta}{2\alpha}} \|u_0\|_{H^s} \ll 1.$$

In particular, we obtain existence, uniqueness, continuous dependence on initial data and regularity for local solutions on $(0, T]$ by a standard argument.

To show that T depends on the initial data only in the H^1 norm, we first obtain the H^1 solution by the above argument (taking $\beta = \frac{1}{2}$) and then notice that

$$\begin{aligned}
& \left\| \int_0^t S(t-t') \mathbb{P}(u(t') \cdot \nabla) u(t') dt' \right\|_{H^s} \\
& \lesssim \int_0^t [\nu(t-t')]^{-\frac{1+\epsilon}{2\alpha}} \| |\nabla|^{s-1} (u \otimes u)(t') \|_{H^{1-\epsilon}} dt' \\
& \lesssim \int_0^t [\nu(t-t')]^{-\frac{1+\epsilon}{2\alpha}} \| |\nabla|^{s-1} u(t') \|_{H^1} \|u(t')\|_{H^{\frac{3}{2}}} dt' \\
& \lesssim \int_0^t [\nu(t-t')]^{-\frac{1+\epsilon}{2\alpha}} (\nu t')^{-\frac{1}{4\alpha}} dt' \|u\|_{X_{T_s}^1} \sup_{0 < t \leq T_s} \|u(t)\|_{H^s} \\
& \lesssim \nu^{-1} (\nu T_s)^{1-\frac{\epsilon+3/2}{2\alpha}} \|u_0\|_{H^1} \sup_{0 < t \leq T_s} \|u(t)\|_{H^s} \leq \frac{1}{2} \sup_{0 < t \leq T_s} \|u(t)\|_{H^s},
\end{aligned}$$

where at the second inequality Lemma 7.1 has been applied with $d = 3$, $s_1 = 1$, $s_2 = \frac{3}{2}$, $s_3 = -1 + \epsilon$ for $0 < \epsilon \ll 1$ (we also divide $|\nabla|^{s-1}$ by a similar argument as in the proof (Case 2) of Lemma 7.1). Here, the above T_s may be smaller than the local existence time T_1 for the H^1 solution (satisfying the lower bound (1.2)). (Note that the s -dependence may come into the estimate when we divide the derivative $|\nabla|^{s-1}$, since $|n|^{s-1} \leq 2^{\min\{s-2, 0\}} (|k|^{s-1} + |n-k|^{s-1})$.) Since u is a solution of (2.1), we have

$$\sup_{0 < t \leq T_s} \|u(t)\|_{H^s} \leq 2 \sup_{0 < t \leq T_s} \|S(t)u_0\|_{H^s} \lesssim \|u_0\|_{H^s}.$$

Repeating this procedure $[T_1/T_s] + 1$ times, we obtain (1.3). In particular, the local existence time for the H^s solution also satisfies the lower bound (1.2).

Finally, we shall prove the global existence for small initial data. For $\frac{3}{4} < \alpha \leq 1$ and $s \geq 1$, we can show that

$$|\langle \nabla(u \otimes u), u \rangle_{H^s}| \leq C(s, \alpha) \|u\|_{H^1} \|u\|_{H^{s+\alpha}}^2.$$

In fact,

$$\begin{aligned}
& \left\| |\nabla|^{s-\alpha} \nabla(fg) \right\|_{L^2} \leq \left\| |n|^{s-\alpha+1} (\widehat{f} * \widehat{g}) \right\|_{\ell^2} \\
& \lesssim_{s,\alpha} \left\| |n|^{s-\alpha+1} \widehat{f} \right\|_{\ell^{\frac{3}{2}-}} \left\| \widehat{g} \right\|_{\ell^{\frac{6}{5}+}} + \left\| \widehat{f} \right\|_{\ell^{\frac{6}{5}+}} \left\| |n|^{s-\alpha+1} \widehat{g} \right\|_{\ell^{\frac{3}{2}-}} \\
& \leq \left(\left\| |n|^{s+\alpha} \widehat{f} \right\|_{\ell^2} \left\| |n| \widehat{g} \right\|_{\ell^2} + \left\| |n| \widehat{f} \right\|_{\ell^2} \left\| |n|^{s+\alpha} \widehat{g} \right\|_{\ell^2} \right) \left\| |n|^{1-2\alpha} \right\|_{\ell^{6-}} \left\| |n|^{-1} \right\|_{\ell^{3+}} \\
& \lesssim_{\alpha} \|f\|_{H^{s+\alpha}} \|g\|_{H^1} + \|f\|_{H^1} \|g\|_{H^{s+\alpha}},
\end{aligned}$$

and then

$$\begin{aligned}
|\langle \nabla(u \otimes u), u \rangle_{H^s}| & \leq \left\| |\nabla|^{s-\alpha} \nabla(u \otimes u) \right\|_{L^2} \left\| |\nabla|^{s+\alpha} u \right\|_{L^2} \\
& \lesssim_{s,\alpha} \|u\|_{H^{s+\alpha}} \|u\|_{H^1} \cdot \|u\|_{H^{s+\alpha}},
\end{aligned}$$

as desired. By skew-symmetry of the Coriolis term in (1.1), the standard H^s -energy estimate with the above inequality implies that

$$(2.3) \quad \frac{d}{dt} \|u(t)\|_{H^s}^2 + 2\nu \|u(t)\|_{H^{s+\alpha}}^2 \leq C(s, \alpha) \|u(t)\|_{H^1} \|u(t)\|_{H^{s+\alpha}}^2.$$

for any smooth solutions $u(t)$ of (1.1). We now assume that the initial data u_0 satisfies the smallness condition

$$\|u_0\|_{H^1} \leq \frac{\nu}{2C(1, \alpha)},$$

and define

$$T_* := \sup \left\{ T \geq 0 \mid \text{the solution } u(t) \text{ exists and } \|u(t)\|_{H^1} \leq \frac{\nu}{C(1, \alpha)} \text{ on } [0, T] \right\}.$$

By the local theory established above, we have $T_* > 0$. Furthermore, (2.3) with $s = 1$ shows that

$$(2.4) \quad \frac{d}{dt} \|u(t)\|_{H^s}^2 + \nu \|u(t)\|_{H^{s+\alpha}}^2 \leq 0, \quad t \in (0, T_*)$$

with $s = 1$. In particular, $\|u(t)\|_{H^1}$ is decreasing and thus $T_* = \infty$. If we further assume that

$$\left(\|u(t)\|_{H^1} \leq \right) \|u_0\|_{H^1} \leq \frac{\nu}{2C(s, \alpha)},$$

then (2.4) holds for this choice of s , which combined with $\|u\|_{H^s} \leq \|u\|_{H^{s+\alpha}}$ implies the exponential decay of $\|u(t)\|_{H^s}$ as $t \rightarrow \infty$.

3. PROPERTIES OF THE NONLINEAR TERM IN THE RESONANT EQUATION

In this section we prove some cancellation properties of the nonlinear term B_R in the resonant equation. We need it in the next section.

Lemma 3.1 (cf. Proposition 6.2 in [7]). *For any divergence-free mean-zero vector field a , we have $\overline{B_R(a_{\text{osc}}, a_{\text{osc}})} = 0$.*

Proof. We first notice that, under $n_3 = 0$ and $k_3 \neq 0$,

$$\omega_{nk(n-k)}^\sigma = \sigma_1 \frac{k_3}{|k|} + \sigma_2 \frac{-k_3}{|n-k|} = 0 \quad \Longleftrightarrow \quad \sigma_1 = \sigma_2, \quad |k| = |n-k|.$$

Hence,

$$\begin{aligned}
& \overline{B_R(a_{\text{osc}}, a_{\text{osc}})}(x) \\
&= i \sum_{\substack{n \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 = 0}} e^{in \cdot x} \sum_{(\sigma, \sigma_3) \in \{\pm\}^2} \sum_{\substack{k \in \mathbb{Z}^3 \setminus \{0\} \\ k_3 \neq 0 \\ |k| = |n-k|}} (\widehat{a}(k) |e^\sigma(k)|) (\widehat{a}(n-k) |e^\sigma(n-k)|) \\
&\quad \cdot \left([e^\sigma(k) \cdot (n-k)] e^\sigma(n-k) \Big| e^{\sigma_3}(n) \right) e^{\sigma_3}(n) \\
&= \frac{i}{2} \sum_{\substack{n \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 = 0}} e^{in \cdot x} \sum_{(\sigma, \sigma_3) \in \{\pm\}^2} \sum_{\substack{k \in \mathbb{Z}^3 \setminus \{0\} \\ k_3 \neq 0 \\ |k| = |n-k|}} (\widehat{a}(k) |e^\sigma(k)|) (\widehat{a}(n-k) |e^\sigma(n-k)|) \\
&\quad \cdot \left([e^\sigma(k) \cdot (n-k)] e^\sigma(n-k) + [e^\sigma(n-k) \cdot k] e^\sigma(k) \Big| e^{\sigma_3}(n) \right) e^{\sigma_3}(n).
\end{aligned}$$

Hence, all we have to do is to show

$$(3.1) \quad \left([e^\sigma(k) \cdot (n-k)] e^\sigma(n-k) + [e^\sigma(n-k) \cdot k] e^\sigma(k) \Big| e^{\sigma_3}(n) \right) = 0$$

for any $\sigma, \sigma_3 \in \{\pm\}$, $n, k \in \mathbb{Z}^3 \setminus \{0\}$ such that $n_3 = 0$, $k_3 \neq 0$, $|k| = |n-k|$.

By the formula in vector analysis, we have

$$\begin{aligned}
& e^\sigma(k) \times [(n-k) \times e^\sigma(n-k)] \\
&= [e^\sigma(k) \cdot e^\sigma(n-k)](n-k) - [e^\sigma(k) \cdot (n-k)] e^\sigma(n-k), \\
& e^\sigma(n-k) \times [k \times e^\sigma(k)] \\
&= [e^\sigma(n-k) \cdot e^\sigma(k)]k - [e^\sigma(n-k) \cdot k] e^\sigma(k).
\end{aligned}$$

By (1.4) and the assumption, we have

$$\begin{aligned}
& e^\sigma(k) \times [(n-k) \times e^\sigma(n-k)] = \sigma i |n-k| [e^\sigma(k) \times e^\sigma(n-k)] \\
&= -\sigma i |k| [e^\sigma(n-k) \times e^\sigma(k)] = -e^\sigma(n-k) \times [k \times e^\sigma(k)].
\end{aligned}$$

Therefore, we have

$$[e^\sigma(k) \cdot (n-k)] e^\sigma(n-k) + [e^\sigma(n-k) \cdot k] e^\sigma(k) = [e^\sigma(k) \cdot e^\sigma(n-k)] n.$$

Since $\widehat{\mathbb{P}}(n)n = 0$, the left-hand side belongs to $(\text{Ran } \widehat{\mathbb{P}}(n))^\perp$, and (3.1) holds. \square

We also recall the following properties of B_R .

Lemma 3.2. *Let $s \geq 0$ and a, b be any divergence-free and mean-zero vector fields. Assume that b is real-valued. Then, we have*

$$\langle B_R(\bar{a}, b_{\text{osc}}), b_{\text{osc}} \rangle_{H^s} = \langle B_R(b_{\text{osc}}, \bar{b}), b_{\text{osc}} \rangle_{H^s} = \langle B_R(a_{\text{osc}}, b_{\text{osc}}), b_{\text{osc}} \rangle_{L^2} = 0.$$

Proof. Let us first consider $B_R(\bar{a}, b_{\text{osc}})$. Similarly to the proof of Lemma 3.1, we see that

$$\begin{aligned} & \langle B_R(\bar{a}, b_{\text{osc}}), b_{\text{osc}} \rangle_{H^s} \\ &= i \sum_{(\sigma_1, \sigma) \in \{\pm\}^2} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 \neq 0 = k_3 \\ |n| = |n-k|}} [\widehat{a}^{\sigma_1}(k) \cdot (n-k)] [\widehat{b}^\sigma(n-k) \cdot e^\sigma(n)^*] [e^\sigma(n) \cdot |n|^{2s} \widehat{b}(n)^*] \\ &= i \sum_{(\sigma_1, \sigma) \in \{\pm\}^2} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 \neq 0 = k_3 \\ |n| = |n-k|}} |n|^{2s} [\widehat{a}^{\sigma_1}(k) \cdot (n-k)] [\widehat{b}^\sigma(n-k) \cdot \widehat{b}^\sigma(n)^*]. \end{aligned}$$

Since b is real-valued, $\widehat{b}(n)^* = \widehat{b}(-n)$ and thus $\widehat{b}^\sigma(n)^* = \widehat{b}^\sigma(-n)$ for any n . By a change of variables $n \mapsto n' := k - n$,

$$\begin{aligned} & \langle B_R(\bar{a}, b_{\text{osc}}), b_{\text{osc}} \rangle_{H^s} \\ &= i \sum_{(\sigma_1, \sigma) \in \{\pm\}^2} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 \neq 0 = k_3 \\ |n| = |n-k|}} |n|^{2s} [\widehat{a}^{\sigma_1}(k) \cdot (n-k)] [\widehat{b}^\sigma(-n) \cdot \widehat{b}^\sigma(k-n)^*] \\ &= i \sum_{(\sigma_1, \sigma) \in \{\pm\}^2} \sum_{\substack{n', k \in \mathbb{Z}^3 \setminus \{0\} \\ n'_3 \neq 0 = k_3 \\ |k-n'| = |n'|}} |k-n'|^{2s} [\widehat{a}^{\sigma_1}(k) \cdot (-n')] [\widehat{b}^\sigma(n'-k) \cdot \widehat{b}^\sigma(n')^*] \\ &= -i \sum_{(\sigma_1, \sigma) \in \{\pm\}^2} \sum_{\substack{n', k \in \mathbb{Z}^3 \setminus \{0\} \\ n'_3 \neq 0 = k_3 \\ |n'| = |n'-k|}} |n'|^{2s} [\widehat{a}^{\sigma_1}(k) \cdot n'] [\widehat{b}^\sigma(n'-k) \cdot \widehat{b}^\sigma(n')^*]. \end{aligned}$$

Since $e^{\sigma_1}(k) \cdot k = 0$, we have

$$\begin{aligned} & 2 \langle B_R(\bar{a}, b_{\text{osc}}), b_{\text{osc}} \rangle_{H^s} \\ &= -i \sum_{(\sigma_1, \sigma) \in \{\pm\}^2} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 \neq 0 = k_3 \\ |n| = |n-k|}} |n|^{2s} [\widehat{a}^{\sigma_1}(k) \cdot k] [\widehat{b}^\sigma(n-k) \cdot \widehat{b}^\sigma(n)^*] = 0. \end{aligned}$$

Next, we consider $B_R(b_{\text{osc}}, \bar{b})$. In a similar manner, by a change of variables $n \mapsto n' := k - n$ and $e^\pm(k) \cdot k = 0$,

$$\begin{aligned} & \langle B_R(b_{\text{osc}}, \bar{b}), b_{\text{osc}} \rangle_{H^s} \\ &= i \sum_{(\sigma, \sigma_2) \in \{\pm\}^2} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 = k_3 \neq 0 \\ |n| = |k|}} |n|^{2s} [\widehat{b}^\sigma(k) \cdot (n-k)] [\widehat{b}^{\sigma_2}(n-k) \cdot \widehat{b}^\sigma(n)^*] \\ &= i \sum_{(\sigma, \sigma_2) \in \{\pm\}^2} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 = k_3 \neq 0 \\ |n| = |k|}} |k|^s |n|^s [\widehat{b}^\sigma(k) \cdot (n-k)] [\widehat{b}^{\sigma_2}(-n) \cdot \widehat{b}^{\sigma_2}(k-n)^*] \end{aligned}$$

$$\begin{aligned}
&= i \sum_{(\sigma, \sigma_2) \in \{\pm\}^2} \sum_{\substack{n', k \in \mathbb{Z}^3 \setminus \{0\} \\ n'_3 = 0 \neq k_3 \\ |k - n'| = |k|}} |k|^s |n' - k|^s [\widehat{b}^\sigma(k) \cdot (-n')] [\widehat{b}^\sigma(n' - k) \cdot \widehat{b}^{\sigma_2}(n')^*] \\
&= -i \sum_{(\sigma, \sigma_2) \in \{\pm\}^2} \sum_{\substack{n', k \in \mathbb{Z}^3 \setminus \{0\} \\ n'_3 = 0 \neq k_3 \\ |k - n'| = |k|}} [|k|^s \widehat{b}^\sigma(k) \cdot (n' - k)] [|n' - k|^s \widehat{b}^\sigma(n' - k) \cdot \widehat{b}^{\sigma_2}(n')^*] \\
&= -\langle B_R(|\nabla|^s b_{\text{osc}}, |\nabla|^s b_{\text{osc}}), b \rangle_{L^2} = 0,
\end{aligned}$$

where we have applied Lemma 3.1 at the last equality.

Finally, since b is real-valued,

$$\begin{aligned}
&\langle B_R(a_{\text{osc}}, b_{\text{osc}}), b_{\text{osc}} \rangle_{L^2} \\
&= i \sum_{(\sigma_1, \sigma_2, \sigma_3) \in \{\pm\}^3} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 k_3 (n_3 - k_3) \neq 0}} [\widehat{a}^{\sigma_1}(k) \cdot (n - k)] [\widehat{b}^{\sigma_2}(n - k) \cdot \widehat{b}^{\sigma_3}(n)^*] \\
&= i \sum_{(\sigma_1, \sigma_2, \sigma_3) \in \{\pm\}^3} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 k_3 (n_3 - k_3) \neq 0}} [\widehat{a}^{\sigma_1}(k) \cdot (n - k)] [\widehat{b}^{\sigma_3}(-n) \cdot \widehat{b}^{\sigma_2}(k - n)^*] \\
&= -i \sum_{(\sigma_1, \sigma'_2, \sigma'_3) \in \{\pm\}^3} \sum_{\substack{n', k \in \mathbb{Z}^3 \setminus \{0\} \\ n'_3 k_3 (n'_3 - k_3) \neq 0}} [\widehat{a}^{\sigma_1}(k) \cdot n'] [\widehat{b}^{\sigma'_2}(n' - k) \cdot \widehat{b}^{\sigma'_3}(n')^*],
\end{aligned}$$

where we have changed the variables as $(\sigma_2, \sigma_3, n) \mapsto (\sigma'_2, \sigma'_3, n') := (\sigma_3, \sigma_2, k - n)$. Therefore,

$$\begin{aligned}
&2 \langle B_R(a_{\text{osc}}, b_{\text{osc}}), b_{\text{osc}} \rangle_{L^2} \\
&= -i \sum_{(\sigma_1, \sigma_2, \sigma_3) \in \{\pm\}^3} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ n_3 k_3 (n_3 - k_3) \neq 0}} [\widehat{a}^{\sigma_1}(k) \cdot k] [\widehat{b}^{\sigma_2}(n - k) \cdot \widehat{b}^{\sigma_3}(n)^*] = 0,
\end{aligned}$$

as desired. \square

4. A PRIORI ESTIMATE AND GLOBAL EXISTENCE FOR THE LIMIT EQUATION

In this section, we shall prove Proposition 1.7. By the local theory, we can solve the limit equation (1.7) in H^1 for a short time and the solution immediately becomes smooth. Therefore, for the global existence, it suffices to derive global a priori H^1 estimate on smooth solutions of (1.7)=(1.8)+(1.9)+(1.10).

Let $\alpha \in (\frac{3}{4}, 1]$, $u_0 \in H^1(\mathbb{T}^3)$ be a divergence-free, mean-zero initial vector field and $M := \|u_0\|_{L^2}$, $E := \|u_0\|_{H^1}$.

4.1. 2D horizontal part (1.8). Note that this is the usual 2D incompressible Navier-Stokes equations. However, when α is strictly less than one, we need to consider equation for the vorticity $\omega = \nabla_h^\perp \overline{U}^h := \partial_{x_1} \overline{U}_2 - \partial_{x_2} \overline{U}_1$:

$$(4.1) \quad \begin{cases} \partial_t \omega + \nu(-\Delta_h)^\alpha \omega + (\overline{U}^h \cdot \nabla^h) \omega = 0, & t > 0, \quad x \in \mathbb{T}^2, \\ \omega|_{t=0} = \nabla_h^\perp \overline{u_0}^h. \end{cases}$$

Note that \overline{U}^h can be recovered from ω by the Biot-Savart law $\overline{U}^h = -(-\Delta_h)^{-1} \nabla_h^\perp \omega$ and

$$\|\omega\|_{H^s} \sim \|\overline{U}^h\|_{H^{s+1}}, \quad s \in \mathbb{R},$$

whenever ω is mean-zero. The standard L^2 -energy estimate for (4.1) yields that

$$\frac{d}{dt} \|\omega(t)\|_{L^2}^2 + 2\nu \|\omega(t)\|_{H^\alpha}^2 \leq 0,$$

or

$$(4.2) \quad \|\overline{U}^h(t)\|_{H^1}^2 + \nu \int_0^t \|\overline{U}^h(t')\|_{H^{1+\alpha}}^2 dt' \leq C \|\overline{u}_0^h\|_{H^1}^2 \leq CE^2, \quad t > 0.$$

4.2. 2D vertical part (1.9). We begin with the easy L^2 -energy estimate for (1.9):

$$(4.3) \quad \|\overline{U}_3(t)\|_{L^2}^2 + 2\nu \int_0^t \|\overline{U}_3(t')\|_{H^\alpha}^2 dt' \leq \|\overline{u}_{0,3}\|_{L^2}^2 \leq M^2, \quad t > 0.$$

For the H^1 -energy estimate, we see that the 2D Sobolev inequality and interpolation argument yields that

$$\begin{aligned} |\langle (\overline{U}^h \cdot \nabla_h) \overline{U}_3, \overline{U}_3 \rangle_{H^1}| &= |\langle \nabla_h \overline{U}^h, \nabla_h \overline{U}_3 \otimes \nabla_h \overline{U}_3 \rangle_{L^2}| \\ &\leq \|\nabla_h \overline{U}^h\|_{L^2} \|\nabla_h \overline{U}_3\|_{L^4}^2 \lesssim \|\overline{U}^h\|_{H^1} \|\overline{U}_3\|_{H^{\frac{3}{2}}}^2 \\ &\leq \|\overline{U}^h\|_{H^1} \|\overline{U}_3\|_{H^{1+\alpha}}^{3-2\alpha} \|\overline{U}_3\|_{H^\alpha}^{2\alpha-1} \\ &\leq \nu \|\overline{U}_3\|_{H^{1+\alpha}}^2 + C\nu^{-\frac{3-2\alpha}{2\alpha-1}} \|\overline{U}^h\|_{H^1}^{\frac{2}{2\alpha-1}} \|\overline{U}_3\|_{H^\alpha}^2. \end{aligned}$$

Note that this estimate is available as long as $\frac{3}{2} \geq \alpha > \frac{1}{2}$. From this we have

$$\frac{d}{dt} \|\overline{U}_3(t)\|_{H^1}^2 + \nu \|\overline{U}_3(t)\|_{H^{1+\alpha}}^2 \leq C\nu^{-\frac{3-2\alpha}{2\alpha-1}} \|\overline{U}^h(t)\|_{H^1}^{\frac{2}{2\alpha-1}} \|\overline{U}_3(t)\|_{H^\alpha}^2, \quad t > 0.$$

Integrating both sides in t and applying (4.2), (4.3), we obtain that

$$\begin{aligned} (4.4) \quad &\|\overline{U}_3(t)\|_{H^1}^2 + \nu \int_0^t \|\overline{U}_3(t')\|_{H^{1+\alpha}}^2 dt' \\ &\leq E^2 + C\nu^{-\frac{3-2\alpha}{2\alpha-1}} \left(\sup_{0 < t' < t} \|\overline{U}^h(t')\|_{H^1}^{\frac{2}{2\alpha-1}} \right) \int_0^t \|\overline{U}_3(t')\|_{H^\alpha}^2 dt' \\ &\leq E^2 + C\nu^{-\frac{2}{2\alpha-1}} E^{\frac{2}{2\alpha-1}} M^2, \quad t > 0. \end{aligned}$$

4.3. Non-trivial resonance part (1.10). By the L^2 energy estimate with Lemma 3.2, we immediately have

$$(4.5) \quad \|U_{\text{osc}}(t)\|_{L^2}^2 + 2\nu \int_0^t \|U_{\text{osc}}(t')\|_{H^\alpha}^2 dt' \leq \|u_{0,\text{osc}}\|_{L^2}^2 \leq M^2, \quad t > 0.$$

The H^1 bound will be obtained from the following lemma:

Lemma 4.1. *For any $\varepsilon > 0$ there exists $C > 0$ such that for any real-valued, divergence-free and mean-zero vector field a , we have*

$$|\langle B_R(a_{\text{osc}}, a_{\text{osc}}), a_{\text{osc}} \rangle_{H^1}| \leq C \|a_{\text{osc}}\|_{H^1}^2 \|a_{\text{osc}}\|_{H^{\frac{3}{2}+\varepsilon}}.$$

This estimate, which is the most important piece in the proof of our result (we will give its proof in the next section), improves in the case of regular (or rational) periodic domains the previous one proved in [3, Theorem 3.1]. The relation to the results of Babin et al. [2, 3] will be discussed in detail in the following subsection.

By Lemmas 3.2 and 4.1, we proceed the H^1 energy estimate as

$$\frac{d}{dt} \|U_{\text{osc}}(t)\|_{H^1}^2 + 2\nu \|U_{\text{osc}}(t)\|_{H^{1+\alpha}}^2 \leq C(\varepsilon) \|U_{\text{osc}}(t)\|_{H^1}^2 \|U_{\text{osc}}(t)\|_{H^{\frac{3}{2}+\varepsilon}}.$$

Let $\alpha \in (\frac{3}{4}, 1]$, and choose $\varepsilon > 0$ so that $2\alpha > \frac{3}{2} + \varepsilon$. By interpolation and Young's inequality,

$$\begin{aligned} C(\varepsilon) \|U_{\text{osc}}\|_{H^1}^2 \|U_{\text{osc}}\|_{H^{\frac{3}{2}+\varepsilon}} &\leq C(\varepsilon) \|U_{\text{osc}}\|_{H^{1+\alpha}}^{\frac{7}{2}-2\alpha+\varepsilon} \|U_{\text{osc}}\|_{L^2} \|U_{\text{osc}}\|_{H^\alpha}^{2\alpha-\frac{3}{2}-\varepsilon} \\ &\leq \nu \|U_{\text{osc}}\|_{H^{1+\alpha}}^2 + C(\varepsilon, \alpha) \nu^{-\frac{7-4\alpha+2\varepsilon}{4\alpha-3-2\varepsilon}} \|U_{\text{osc}}\|_{L^2}^{\frac{4}{4\alpha-3-2\varepsilon}} \|U_{\text{osc}}\|_{H^\alpha}^2, \end{aligned}$$

and hence,

$$\begin{aligned} \frac{d}{dt} \|U_{\text{osc}}(t)\|_{H^1}^2 + \nu \|U_{\text{osc}}(t)\|_{H^{1+\alpha}}^2 \\ \leq C(\varepsilon, \alpha) \nu^{-\frac{7-4\alpha+2\varepsilon}{4\alpha-3-2\varepsilon}} \|U_{\text{osc}}(t)\|_{L^2}^{\frac{4}{4\alpha-3-2\varepsilon}} \|U_{\text{osc}}(t)\|_{H^\alpha}^2, \quad t > 0. \end{aligned}$$

Integrating both sides in t and applying (4.5), we obtain that

$$\begin{aligned} (4.6) \quad &\|U_{\text{osc}}(t)\|_{H^1}^2 + \nu \int_0^t \|U_{\text{osc}}(t')\|_{H^{1+\alpha}}^2 dt' \\ &\leq E^2 + C(\varepsilon, \alpha) \nu^{-\frac{7-4\alpha+2\varepsilon}{4\alpha-3-2\varepsilon}} \left(\sup_{0 < t' < t} \|U_{\text{osc}}(t')\|_{L^2}^{\frac{4}{4\alpha-3-2\varepsilon}} \right) \int_0^t \|U_{\text{osc}}(t')\|_{H^\alpha}^2 dt' \\ &\leq E^2 + C(\varepsilon, \alpha) \nu^{-\frac{4}{4\alpha-(3+2\varepsilon)}} M^{2+\frac{4}{4\alpha-(3+2\varepsilon)}}, \quad t > 0. \end{aligned}$$

Combining (4.2), (4.4) and (4.6), we obtain a global H^1 -a priori estimate on the solution $U(t)$ of the limit equation (1.7) as

$$\begin{aligned} (4.7) \quad &\|U(t)\|_{H^1}^2 + \nu \int_0^t \|U(t')\|_{H^{1+\alpha}}^2 dt' \\ &\leq CE^2 + C\nu^{-\frac{2}{2\alpha-1}} M^2 E^{\frac{2}{2\alpha-1}} + C(\varepsilon, \alpha) \nu^{-\frac{4}{4\alpha-(3+2\varepsilon)}} M^{2+\frac{4}{4\alpha-(3+2\varepsilon)}} \\ &\lesssim_{\alpha, \varepsilon} E^2 \left\{ 1 + \left(\frac{E}{\nu} \right)^{\frac{4}{4\alpha-(3+2\varepsilon)}} \right\}, \end{aligned}$$

where $0 < \varepsilon < 2\alpha - \frac{3}{2}$ and we have used $M \leq E$, $0 \leq \frac{2}{2\alpha-1} \leq \frac{4}{4\alpha-(3+2\varepsilon)}$. This is enough to show the existence of global regular solutions $U(t)$ of (1.7).

Note that the last line of (4.7) is constant in t . This immediately implies that the solution $U(t)$ is bounded in H^1 . Moreover, since it is easy to see $\langle B_R(U, U), U \rangle_{L^2} = 0$ for real-valued divergence-free and mean-zero vector field U , we obtain the L^2 -energy estimate for (1.7) as

$$\|U(t)\|_{L^2}^2 + 2\nu \int_0^t \|U(t')\|_{H^\alpha}^2 dt' \leq M^2,$$

which is interpolated with (4.7) to yield

$$\int_0^\infty \|U(t')\|_{H^1}^2 dt' \lesssim_{\alpha, \varepsilon} \frac{E^2}{\nu} \left\{ 1 + \left(\frac{E}{\nu} \right)^{\frac{4(1-\alpha)}{4\alpha-(3+2\varepsilon)}} \right\}.$$

In particular, it holds that

$$(4.8) \quad \forall \eta > 0, \quad 0 < \exists t_\eta \lesssim_{\alpha, \varepsilon} \eta^{-2} \frac{E^2}{\nu} \left\{ 1 + \left(\frac{E}{\nu} \right)^{\frac{4(1-\alpha)}{4\alpha - (3+2\varepsilon)}} \right\} \quad \text{s.t.} \quad \|U(t_\eta)\|_{H^1} \leq \eta.$$

This fact will be important when we estimate the difference between $U(t)$ and the solution of (1.5). By the result of small-data global existence (similar to Theorem 1.2), we see that, for any $s \geq 1$, the H^s norm of the solution $U(t)$ decays exponentially after some time $T_C = T_C(s, \alpha, \varepsilon, \nu, E) > 0$.

We have thus established Proposition 1.7, up to the proof of the key Lemma 4.1.

4.4. Remarks. Lemma 4.1 should be compared to the previous one by Babin et al. ([3, Theorem 3.1]). Let us recall some results in [2, 3].

Let $a_2, a_3 > 0$ be positive numbers and consider the problem on the torus $\mathbf{T}_a^3 := [0, 2\pi) \times [0, 2\pi a_2) \times [0, 2\pi a_3)$. (We may always assume the period in the x_1 direction to be equal to 2π by rescaling the torus.) The Fourier series is defined by

$$u(x) = \sum_{n \in \mathbb{Z}^3} \hat{u}(n) e^{i\tilde{n} \cdot x}, \quad \hat{u}(n) := \frac{1}{(2\pi)^3 a_2 a_3} \int_{\mathbf{T}_a^3} u(x) e^{-i\tilde{n} \cdot x} dx,$$

where $\tilde{n} = (n_1, \frac{n_2}{a_2}, \frac{n_3}{a_3})$, and Sobolev spaces $H^s(\mathbf{T}_a^3)$ is defined in a natural way. The eigenvalues of the matrix $\hat{\mathbb{P}}(n)J\hat{\mathbb{P}}(n)$ are $\pm i\tilde{n}_3/|\tilde{n}|$, and the nontrivial resonance condition can be written as

$$\exists \sigma \in \{\pm\}^3; \quad \sigma_1 \frac{k_3}{|\tilde{k}|} + \sigma_2 \frac{m_3}{|\tilde{m}|} = \sigma_3 \frac{n_3}{|\tilde{n}|}$$

with $k_3 m_3 n_3 \neq 0$ and the convolution condition

$$(4.9) \quad k + m = n.$$

This is equivalent to

$$\begin{aligned} 0 &= \prod_{\sigma_1, \sigma_2 \in \{\pm\}} \left(\sigma_1 \frac{k_3}{|\tilde{k}|} + \sigma_2 \frac{m_3}{|\tilde{m}|} - \frac{n_3}{|\tilde{n}|} \right) \\ &= \left(\frac{k_3^2}{|\tilde{k}|^2} + \frac{m_3^2}{|\tilde{m}|^2} - \frac{n_3^2}{|\tilde{n}|^2} \right)^2 - 4 \frac{k_3^2 m_3^2}{|\tilde{k}|^2 |\tilde{m}|^2}, \end{aligned}$$

or

$$(4.10) \quad P(k, m, n) = 0$$

with

$$P(k, m, n) := (k_3^2 |\tilde{m}|^2 |\tilde{n}|^2 + m_3^2 |\tilde{k}|^2 |\tilde{n}|^2 - n_3^2 |\tilde{k}|^2 |\tilde{m}|^2)^2 - 4k_3^2 m_3^2 |\tilde{k}|^2 |\tilde{m}|^2 |\tilde{n}|^4.$$

If we write $\theta_2 := a_2^{-2}$ and $\theta_3 := a_3^{-2}$, then

$$|\tilde{n}|^2 = n_1^2 + \theta_2 n_2^2 + \theta_3 n_3^2$$

and similarly for k, m , and thus $P(k, m, n)$ is a polynomial of degree 4 in θ_2, θ_3 and the coefficient of θ_3^4 ($= -3k_3^4 m_3^4 n_3^4$) does not vanish whenever $k_3 m_3 n_3 \neq 0$.

On the other hand, (4.10) determines algebraic curves $\Gamma(k, m, n)$ in the (θ_2, θ_3) -plane parametrized by k, m, n . We see that the equation (4.10) (i.e. the curve

$\Gamma(k, m, n)$ with the convolution condition (4.9) is invariant under dilations, reflections and permutations:

$$\begin{aligned} (k, m, n) &\mapsto (\lambda k, \lambda m, \lambda n), \quad \lambda \in \mathbb{R} \setminus \{0\}; \\ (k, m, n) &\mapsto (Rk, Rm, Rn), \quad R \in \left\{ \text{Id}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}; \\ (k, m, n) &\mapsto (S(k), S(m), -S(-n)), \quad S: \text{any permutation of } \{k, m, -n\}. \end{aligned}$$

Hence, if we write L_n to denote the straight line in the Fourier space through the origin and n , then the curve $\Gamma(k, m, n)$ depends only on (unordered) triplets $\{L_k, L_m, L_n\}$ and does not depend on their (simultaneous) reflections.

It was shown in [3, Section 4] that if we assume $k + m = n$, $k_3 m_3 n_3 \neq 0$ and that the curve $\Gamma(k, m, n)$ intersects with the first quadrant of the (θ_2, θ_3) -plane, then the curve is represented as the graph of a function $\theta_3 = \phi_3(\theta_2)$ on the first quadrant, and moreover,

- if $(k_3 m_2 - k_2 m_3)(k_1 m_3 - k_3 m_1)(k_1 m_2 - k_2 m_1) = 0$, then the curve is reduced to a straight line;
- otherwise, the curve is irreducible. In this case, the coincidence of two such curves $\Gamma(k, m, n) = \Gamma(k', m', n')$ implies the coincidence of the sets

$$\begin{aligned} \left\{ \frac{k_1^2}{k_3^2}, \frac{m_1^2}{m_3^2}, \frac{n_1^2}{n_3^2} \right\} &= \left\{ \frac{(k'_1)^2}{(k'_3)^2}, \frac{(m'_1)^2}{(m'_3)^2}, \frac{(n'_1)^2}{(n'_3)^2} \right\}, \\ \left\{ \frac{k_2^2}{k_3^2}, \frac{m_2^2}{m_3^2}, \frac{n_2^2}{n_3^2} \right\} &= \left\{ \frac{(k'_2)^2}{(k'_3)^2}, \frac{(m'_2)^2}{(m'_3)^2}, \frac{(n'_2)^2}{(n'_3)^2} \right\}. \end{aligned}$$

Based on these facts, the numbers N_r, N_{ir} were defined for given θ_2, θ_3 as follows:

$$\begin{aligned} N_r(\theta_2, \theta_3) &:= \# \left\{ \Gamma(k, m, n) \mid \begin{array}{l} k_3 m_3 n_3 \neq 0, k + m = n, (\theta_2, \theta_3) \in \Gamma(k, m, n) \\ \Gamma: \text{straight line} \end{array} \right\}, \\ N_{ir}(\theta_2, \theta_3) &:= \sup_L \# \left\{ \Gamma(k, m, n) \mid \begin{array}{l} k_3 m_3 n_3 \neq 0, k + m = n, (\theta_2, \theta_3) \in \Gamma(k, m, n) \\ \Gamma: \text{irreducible curve s.t. } RL \in \{L_k, L_m, L_n\} \end{array} \right\}, \end{aligned}$$

where L ranges over all the lines through the origin and R denotes reflection symmetries.

Babin et al. [2, 3] studied the global regularity for (1.1) with $\alpha = 1$ in general periodic domains and made a refined analysis for domains with $N_{ir}(\theta_2, \theta_3) < \infty$. Their result on the estimate of non-trivial resonant part can be rewritten with our notations as follows:

- (i) $N_r = N_{ir} = 0$ holds for almost all (θ_2, θ_3) . In this case, non-trivial resonances do not occur; namely, $\langle B_R(a_{\text{osc}}, a_{\text{osc}}), a_{\text{osc}} \rangle_{H^1} \equiv 0$. One has uniform-in- ν^{-1} a priori bound

$$\|U_{\text{osc}}(t)\|_{H^1}^2 \leq \|U_{\text{osc}}(0)\|_{H^1}^2$$

for solutions $U_{\text{osc}}(t)$ of (1.10) with $\alpha = 1$, which even implies long time existence under fast rotation for inviscid flow, as shown in [2].

- (ii) In the case where $N_r = 0$ and $0 < N_{ir} < \infty$, non-trivial resonances do exist but are finitely many, i.e. “0D like”. One obtains an a priori bound

$$\|U_{\text{osc}}(t)\|_{H^1}^2 \leq \|U_{\text{osc}}(0)\|_{H^1}^2 + \frac{CN_{ir}}{\nu^2} \|U_{\text{osc}}(0)\|_{L^2}^4.$$

- (iii) In the case where $0 < N_r \leq \infty$ and $0 \leq N_{ir} < \infty$, non-trivial resonance is “1D like”, and the a priori bound obtained is

$$\|U_{\text{osc}}(t)\|_{H^1}^2 \leq \|U_{\text{osc}}(0)\|_{H^1}^2 + \frac{CN_{ir}}{\nu^2} \|U_{\text{osc}}(0)\|_{L^2}^4 + \frac{C}{\nu^4} \|U_{\text{osc}}(0)\|_{L^2}^6.$$

- (iv) In general, including the “worst case” of $N_{ir} = \infty$, one has the bound

$$\|U_{\text{osc}}(t)\|_{H^1}^2 \leq \|U_{\text{osc}}(0)\|_{H^1}^2 \exp \left[\frac{C}{\nu^2} \|U_{\text{osc}}(0)\|_{L^2}^2 \right].$$

In (ii) and (iii), one gets polynomial-in- ν^{-1} a priori bound similar to (4.6) shown above. In particular, one can also show global regularity for (1.1) with α less than 1 on these domains (optimal range for α may depend on whether $N_r = 0$ or not). However, in the case (iv) the estimates in [3] are not sufficient to treat the fractional Laplacian with $\alpha < 1$.

The general estimate in [3, Theorem 3.1] was based on an observation that the total number of non-trivial resonant frequency triplets is “2D like”, though the interactions are genuinely 3D. This claim seems rather obvious (and actually was verified by a very elementary argument) because the resonance constraint represented by one nontrivial equality should reduce possibility by at least one dimension. Now, it is also natural to expect that the non-trivial resonance is in fact much “rarer” event, since the resonance relation defines a surface of *nonzero curvature* in frequency space. As we will see in the proof of Lemma 4.1, this kind of heuristics can be justified by a combinatorial argument in the case of regular or rational domains. (Lemma 4.1 says that the non-trivial resonances are actually “ $(1 + \varepsilon)$ D like”.) Such a combinatorial argument is a standard tool in the study of periodic nonlinear dispersive equations (see e.g. [5, 6, 18]), while it seems new in the context of equations of rotating fluids in a periodic domain. We also note that it is not clear whether the “ $(1 + \varepsilon)$ D like” estimate is optimal or not.

Finally, we claim that the regular (i.e. $\theta_2 = \theta_3 = 1$) domains considered in this paper are in fact included in the $N_{ir} = \infty$ case. However, it is hard in general to determine the precise values of N_r and N_{ir} for given (θ_2, θ_3) .

Lemma 4.2. $N_{ir}(1, 1) = \infty$.

Proof. We will prove it by constructing infinitely many triplets $\{(k_j, m_j, n_j)\}_{j \geq 1}$ satisfying $k_{j,3}m_{j,3}n_{j,3} \neq 0$, $k_j + m_j = n_j$, $n_j \in L_{(1,0,1)}$ and generating mutually different irreducible curves $\Gamma(k_j, m_j, n_j)$ in the (θ_2, θ_3) -plane passing through $(1, 1)$.

Let us look for a triplet (k, m, n) of the form

$$k = (x, 1, y), \quad m = (y, -1, x), \quad n = k + m = (x + y, 0, x + y); \quad x, y \in \mathbb{Z}.$$

To ensure that $(1, 1) \in \Gamma(k, m, n)$, we impose the following condition:

$$\frac{k_3}{|k|} + \frac{m_3}{|m|} = \frac{n_3}{|n|} \text{ or } -\frac{n_3}{|n|}, \quad \text{i.e.,} \quad \frac{|x+y|}{\sqrt{x^2+y^2+1}} = \frac{1}{\sqrt{2}}.$$

This is equivalent to

$$(4.11) \quad x^2 + 4xy + y^2 = 1.$$

Setting $X := x + 2y$, we get

$$(4.12) \quad X^2 - 3y^2 = 1.$$

This is one of Pell's equations known to have infinitely many integer solutions. In fact, $(X, y) = (X_1, y_1) := (2, 1)$ is a solution, and by the theory of Pell's equation,

$$(X_j, y_j) \in \mathbb{N}^2 \quad \text{defined by} \quad X_j + y_j\sqrt{3} = (X_1 + y_1\sqrt{3})^j, \quad j = 1, 2, 3, \dots$$

are all solutions of (4.12). Since

$$X_{j+1} + y_{j+1}\sqrt{3} = (X_j + y_j\sqrt{3})(2 + \sqrt{3}) = (2X_j + 3y_j) + (X_j + 2y_j)\sqrt{3},$$

the corresponding solutions (x_j, y_j) of (4.11) with $X_j = x_j + 2y_j$ satisfies the recurrence relations

$$\begin{aligned} y_{j+1} &= X_j + 2y_j = x_j + 4y_j, \\ x_{j+1} &= X_{j+1} - 2y_{j+1} = (2X_j + 3y_j) - 2(X_j + 2y_j) = -y_j, \end{aligned}$$

with $(x_1, y_1) = (0, 1)$. Then, $(\tilde{x}_j, \tilde{y}_j) := (-1)^{j-1}(x_j, y_j)$ is also a solution of (4.11) satisfying

$$\tilde{x}_{j+1} = \tilde{y}_j, \quad \tilde{y}_{j+1} = -\tilde{x}_j - 4\tilde{y}_j = -\tilde{y}_{j-1} - 4\tilde{y}_j; \quad (\tilde{x}_1, \tilde{y}_1) = (0, 1).$$

Therefore, we have $(\tilde{x}_j, \tilde{y}_j) = (a_{j-1}, a_j)$ with the sequence $\{a_j\}_{j \geq 0}$ defined by

$$(4.13) \quad a_0 = 0, \quad a_1 = 1, \quad a_{j+2} + 4a_{j+1} + a_j = 0 \quad (j \geq 0).$$

So far, we have obtained a sequence of triplets $\{(k_j, m_j, n_j)\}_{j \geq 1}$,

$$k_j = (a_j, 1, a_{j+1}), \quad m_j = (a_{j+1}, -1, a_j), \quad n_j = (a_j + a_{j+1}, 0, a_j + a_{j+1}),$$

for which $k_j + m_j = n_j$, $n_j \in L_{(1,0,1)}$, and by the above construction of $\{a_j\}$, the curve $\Gamma(k_j, m_j, n_j)$ passes through $(1, 1)$. (Given the sequence $\{a_j\}$ defined by (4.13), one can also show directly without using the theory of Pell's equation that $(x, y) = (a_j, a_{j+1})$ satisfies (4.11), and hence $(1, 1) \in \Gamma(k_j, m_j, n_j)$, by an induction on j .)

It remains to check $k_{j,3}m_{j,3}n_{j,3} \neq 0$ and that $\{\Gamma(k_j, m_j, n_j)\}_{j \geq 1}$ are mutually different irreducible curves. By (4.13), we see that $|a_{j+1}| \geq 3|a_j| + 1$ for any $j \geq 0$; in fact,

$$\begin{aligned} |a_{j+1}| - 3|a_j| &= |4a_j + a_{j-1}| - 3|a_j| \geq (|a_j| - 3|a_{j-1}|) + 2|a_{j-1}| \\ &\geq |a_j| - 3|a_{j-1}| \geq \dots \geq |a_1| - 3|a_0| = 1. \end{aligned}$$

In particular, it holds that

$$|a_j| \neq |a_{j'}| \quad (j \neq j'); \quad a_j \neq 0 \quad (j \geq 1).$$

This ensures that $k_{j,3}m_{j,3}n_{j,3} \neq 0$. Moreover, we have

$$\begin{aligned} &(k_{j,3}m_{j,2} - k_{j,2}m_{j,3})(k_{j,1}m_{j,3} - k_{j,3}m_{j,1})(k_{j,1}m_{j,2} - k_{j,2}m_{j,1}) \\ &= (a_j + a_{j+1})^3(a_j - a_{j+1}) \neq 0, \end{aligned}$$

which shows that $\Gamma(k_j, m_j, n_j)$ is irreducible. Finally, the sets

$$\left\{ \frac{k_{j,2}^2}{k_{j,3}^2}, \frac{m_{j,2}^2}{m_{j,3}^2}, \frac{n_{j,2}^2}{n_{j,3}^2} \right\} = \{a_{j+1}^{-2}, a_j^{-2}, 0\}, \quad j \geq 1$$

are mutually different, and so are the curves $\Gamma(k_j, m_j, n_j)$. \square

5. THE KEY ESTIMATE ON THE NON-TRIVIAL RESONANT PART.

Here, we shall give a proof of Lemma 4.1. Recall $\omega_{nkm}^\sigma = \sigma_1 \frac{k_1}{|k|} + \sigma_2 \frac{m_3}{|m|} + \sigma_3 \frac{n_3}{|n|}$. Define the set of non-trivial resonant frequencies $K^* \subset (\mathbb{Z}^3)^3$ as

$$K^* := \{(n, k, m) \in (\mathbb{Z}^3)^3 \mid k + m = n, k_3 m_3 n_3 \neq 0, \omega_{nkm}^\sigma = 0 \text{ for some } \sigma \in \{\pm\}^3\}.$$

We also use the notation $\mathbb{Z}_*^3 := \{n \in \mathbb{Z}^3 \mid n_3 \neq 0\}$.

The following lemma is crucial in the proof:

Lemma 5.1. *Let $L \geq 1$. Then, for any $\varepsilon > 0$ we have*

$$\sup_{n \in \mathbb{Z}_*^3} \#\{(k, m) \in (\mathbb{Z}_*^3)^2 \mid (n, k, m) \in K^*, |k| \leq L\} \leq CL^{1+\varepsilon},$$

where the constant $C > 0$ depends only on ε .

Remark 5.2. In [3, 7], they used the following estimate instead of the above:

$$\sup_{n \in \mathbb{Z}_*^3} \#\{(k, m) \in (\mathbb{Z}_*^3)^2 \mid (n, k, m) \in K^*, |k| \leq L\} \leq CL^2.$$

This estimate follows easily from the fact that the resonant constraint $\omega_{nk(n-k)}^\sigma = 0$ determines an algebraic equation in k_3 of order 8 for each fixed n and (k_1, k_2) . In particular, this estimate requires no combinatorial argument, and hence it holds for any periodic domains. On the other hand, the following proof of Lemma 5.1 is available only for regular (or rational) domains.

Proof of Lemma 5.1. We rely on the well-known lemma in elementary number theory:

Lemma 5.3 (divisor bound, cf. Theorems 278 and 315 in [17]). *For any $\varepsilon > 0$ there exists $C > 0$ such that the following estimates hold for any positive integer N .*

- (i) $\#\{\text{divisors of } N\} \leq CN^\varepsilon$.
- (ii) $\#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = N\} \leq CN^\varepsilon$.

We focus on the case $\sigma = (+, +, +)$; a similar proof applies for other cases.

For given $n, k, m \in \mathbb{Z}_*^3$, positive integers $\nu, \kappa, \mu, d_n, d_k, d_m$ are uniquely determined so that

$$|n| = \nu\sqrt{d_n}, \quad |k| = \kappa\sqrt{d_k}, \quad |m| = \mu\sqrt{d_m}, \quad d_n, d_k, d_m : \text{square-free.}$$

We first see that $d_n = d_k = d_m$ if $\omega_{nkm}^\sigma = 0$. In fact, we have

$$\frac{n_3^2}{|n|^2} - \frac{2n_3k_3}{|n||k|} + \frac{k_3^2}{|k|^2} = \frac{m_3^2}{|m|^2},$$

hence $|n||k| = \nu\kappa\sqrt{d_n d_k}$ must be in \mathbb{Q} , which means $d_n = d_k$ since both d_n and d_k are square-free. Similarly we have $d_n = d_m$. Therefore, we may write uniquely as

$$|n| = \nu\sqrt{d}, \quad |k| = \kappa\sqrt{d}, \quad |m| = \mu\sqrt{d}, \quad d : \text{square-free.}$$

Now, we fix $n \in \mathbb{Z}_*^3$ and count the number of $k \in \mathbb{Z}_*^3$ such that $n_3 \neq k_3$, $\omega_{nk(n-k)}^\sigma = 0$ and $|k| \leq L$. (Note that ν, d are determined once n is fixed.) First, there are at most $2L$ choices for k_3 , since $|k| \leq L$.

Next we fix k_3 , so that $n_3 - k_3$ is also fixed. We shall prove that there are at most $O(L^\varepsilon)$ choices for κ . Before proving it, we note that there are at most $O(L^{2\varepsilon})$ choices for (k_1, k_2) after fixing κ , because $k_1^2 + k_2^2 = |k|^2 - k_3^2 = \kappa^2 d - k_3^2 =: N$ is

a fixed positive integer and we can apply Lemma 5.3 (ii), noticing $N \leq |k|^2 \leq L^2$. These estimates imply the desired bound on the number of k 's. More precisely, we just multiply all possibilities; $O(L)$ for k_3 , $O(L^\varepsilon)$ for κ and $O(L^{2\varepsilon})$ for (k_1, k_2) .

Now we estimate the total number of possible κ 's for fixed n and k_3 , considering the following three cases separately.

(I) $|n| \lesssim L^6$: We see that

$$\begin{aligned} \omega_{nk(n-k)}^\sigma = 0 &\iff \frac{k_3}{\kappa} + \frac{n_3 - k_3}{\mu} = \frac{n_3}{\nu} \\ &\iff (n_3\kappa - k_3\nu)(n_3\mu - (n_3 - k_3)\nu) = k_3(n_3 - k_3)\nu^2. \end{aligned}$$

Therefore, $n_3\kappa - k_3\nu \in \mathbb{Z}$ divides the fixed integer $k_3(n_3 - k_3)\nu^2$ of size $O(L^{1+6+6\cdot 2})$. By Lemma 5.3 (i), there are at most $O(L^\varepsilon)$ choices for $n_3\kappa - k_3\nu \in \mathbb{Z}$. This implies that there are at most $O(L^\varepsilon)$ possibilities for κ , because n_3, k_3, ν are all already determined.

(II) $|n| \gg L^6$, $|n_3| \lesssim |n|^{1/2}$: We show that this case does not occur. In fact, it holds that $|n - k| \sim |n|$ and $|k| \leq L \ll |n|^{1/2}$ in this case. We have

$$\frac{1}{L} \leq \frac{1}{|k|} \leq \left| \frac{k_3}{|k|} \right| \leq \left| \frac{n_3}{|n|} \right| + \left| \frac{n_3 - k_3}{|n - k|} \right| \lesssim \frac{|n|^{1/2}}{|n|} = \frac{1}{|n|^{1/2}},$$

which is not consistent with $|n| \gg L^6$.

(III) $|n| \gg L^6$, $|n_3| \gg |n|^{1/2}$: In this case we show that there are at most four choices for κ 's. Suppose for contradiction that there are five possibilities for κ . Since $(\kappa, \mu) \in \mathbb{N}^2$ satisfies

$$\left(\kappa - \frac{k_3\nu}{n_3} \right) \left(\mu - \frac{(n_3 - k_3)\nu}{n_3} \right) = \frac{k_3(n_3 - k_3)\nu^2}{n_3^2},$$

at least three different (non-collinear) points $P_j := (\kappa_j, \mu_j) \in \mathbb{Z}^2$ ($j = 1, 2, 3$) are on the same component of the fixed hyperbola (in this order):

$$\begin{aligned} &\{(x, y) \in \mathbb{R}^2 : (x - a)(y - b) = M\}, \\ &a = \frac{k_3\nu}{n_3}, \quad b = \frac{(n_3 - k_3)\nu}{n_3}, \quad M = \frac{k_3(n_3 - k_3)\nu^2}{n_3^2}. \end{aligned}$$

Now, an elementary calculation shows: For fixed $a, b, M \in \mathbb{R}$, the area S of the region surrounded by (one component of) hyperbola $(x - a)(y - b) = M$ and a chord of length Λ is at most

$$(5.1) \quad S = O(\Lambda^3 / \sqrt{|M|}), \quad \text{whenever } \sqrt{|M|} \gg \Lambda.$$

To prove this, we may assume that $a = b = 0$ and $M > 0$ without loss of generality. Let $x_0 > 0$ and define $S(x_0)$ as the area of the region surrounded by the hyperbola and the segment between two points $(x_0, \frac{M}{x_0})$, $(x_0 + \Lambda, \frac{M}{x_0 + \Lambda})$. By symmetry of hyperbolic curves on diagonal lines, S is bounded by $\sup_{x_0 \geq \sqrt{M} - \Lambda} S(x_0)$ if $\sqrt{M} > \Lambda$.

Now, for $x_0 \geq \sqrt{M} - \Lambda$, noticing that $\eta := \frac{\Lambda}{x_0} \ll 1$ if $\sqrt{M} \gg \Lambda$, we have

$$\begin{aligned} S(x_0) &= \frac{\Lambda}{2} \left(\frac{M}{x_0} + \frac{M}{x_0 + \Lambda} \right) - \int_{x_0}^{x_0 + \Lambda} \frac{M}{x} dx \\ &= M \left[\frac{\eta}{2} + \frac{\eta}{2(1 + \eta)} - \log(1 + \eta) \right] \\ &= M \left[\frac{\eta}{2} + \frac{\eta}{2}(1 - \eta + \eta^2 + O(\eta^3)) - (\eta - \frac{\eta^2}{2} + \frac{\eta^3}{3} + O(\eta^4)) \right] \\ &= M \left(\frac{\eta^3}{6} + O(\eta^4) \right). \end{aligned}$$

By the above estimate, we have (5.1).

In our case, (κ, μ) is already confined to $[0, L/\sqrt{d}] \times [\nu - L/\sqrt{d}, \nu + L/\sqrt{d}]$, so the length of the segment $P_1 P_3$ is at most $\sqrt{5}L/\sqrt{d}$. Since $|n_3| \gg L \geq |k_3|$ and $|n| \gg L^2$, we have

$$\left| \frac{k_3(n_3 - k_3)\nu^2}{n_3^2} \right| = |k_3| \left| \frac{n_3 - k_3}{n_3} \right| \left| \frac{\nu^2}{n_3} \right| \gtrsim \frac{\nu^2}{|n_3|} = \frac{|n|}{|n_3|} \frac{|n|}{d} \geq \frac{|n|}{d} \gg \left(\frac{L}{\sqrt{d}} \right)^2.$$

Hence, we can apply (5.1) with $M \gtrsim |n|/d$ and $\Lambda \lesssim L/\sqrt{d}$ to show that the area of the triangle $P_1 P_2 P_3$ is bounded by

$$C \left(\frac{L}{\sqrt{d}} \right)^3 \left(\frac{d}{|n|} \right)^{1/2} \lesssim \frac{L^3}{|n|^{1/2}} \ll 1,$$

where we have used the assumption $|n| \gg L^6$. This is a contradiction, because the area of a non-degenerate lattice triangle is bounded from below by $\frac{1}{2}$. Therefore, the case (III) has been proved.

This completes the proof of Lemma 5.1. \square

Lemma 5.1 and a standard argument using the Littlewood-Paley decomposition imply the following Sobolev estimate. To show Lemma 4.1, we just apply the following lemma with $\alpha = \beta = 0$ and $\gamma = \frac{1}{2} + \varepsilon$, noticing that

$$\langle B_R(a_{\text{osc}}, a_{\text{osc}}), a_{\text{osc}} \rangle_{H^1} = \langle B_R(\nabla a_{\text{osc}}, a_{\text{osc}}), \nabla a_{\text{osc}} \rangle_{L^2}.$$

Lemma 5.4. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy $\alpha + \beta + \gamma \geq \max\{\alpha, \beta, \gamma\}$ and $\alpha + \beta + \gamma > \frac{1}{2}$. Then, the following estimate holds.*

$$\left| \sum_{(n, k, m) \in K^*} \widehat{f}(k) \widehat{g}(m) \widehat{h}(n) \right| \lesssim \|f\|_{H^\alpha} \|g\|_{H^\beta} \|h\|_{H^\gamma}.$$

Proof. By symmetry of K^* (considering $-n$ instead of n), we may restrict the summation in the left-hand side onto the frequencies satisfying $|k| \geq |m| \geq |n|$.

We define the dyadic set $\Sigma_j := \{k \in \mathbb{Z}^3 : 2^j \leq |k| < 2^{j+1}\}$ for $j = 0, 1, 2, \dots$ and decompose mean-zero f as $f = \sum_{j \geq 0} f_j$ with $\widehat{f}_j := \widehat{f} \chi_{\Sigma_j}$, and similarly for g and h . Note that $\|f\|_{H^\alpha} = (\sum_{j \geq 0} \|f_j\|_{H^\alpha}^2)^{1/2}$. Since $k + m = n$ and $|k| \geq |m| \geq |n|$ implies $|k| \leq 2|m|$, it holds that

$$S := \left| \sum_{\substack{(n, k, m) \in K^* \\ |k| \geq |m| \geq |n|}} \widehat{f}(k) \widehat{g}(m) \widehat{h}(n) \right| \leq \sum_{j \geq 0} \sum_{j' = j, j+1} \sum_{0 \leq l \leq j} \sum_{(n, k, m) \in K^*} |\widehat{f}_{j'}(k) \widehat{g}_j(m) \widehat{h}_l(n)|.$$

Take $\varepsilon, \delta > 0$ so that $\frac{1}{2}(1+\varepsilon) + \delta \leq \alpha + \beta + \gamma$. By the Cauchy-Schwarz inequality and Lemma 5.1, we have

$$\begin{aligned}
& 2^{\alpha j' + \beta j + \gamma l} \sum_{(n,k,m) \in K^*} |\widehat{f}_{j'}(k) \widehat{g}_j(m) \widehat{h}_l(n)| \\
& \leq \left(\sum_{k \in \mathbb{Z}^3} 2^{2\alpha j'} |\widehat{f}_{j'}(k)|^2 \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^3} \left(\sum_{\substack{m,n \in \mathbb{Z}^3 \\ (n,k,m) \in K^*}} 2^{\beta j} |\widehat{g}_j(m)| \cdot 2^{\gamma l} |\widehat{h}_l(n)| \right)^2 \right)^{1/2} \\
& \lesssim 2^{\frac{1}{2}(1+\varepsilon)l} \left(\sum_{k \in \mathbb{Z}^3} 2^{2\alpha j'} |\widehat{f}_{j'}(k)|^2 \right)^{1/2} \left(\sum_{k \in \mathbb{Z}^3} \sum_{\substack{m,n \in \mathbb{Z}^3 \\ (n,k,m) \in K^*}} 2^{2\beta j} |\widehat{g}_j(m)|^2 \cdot 2^{2\gamma l} |\widehat{h}_l(n)|^2 \right)^{1/2} \\
& \lesssim 2^{\frac{1}{2}(1+\varepsilon)l} \|f_{j'}\|_{H^\alpha} \|g_j\|_{H^\beta} \|h_l\|_{H^\gamma},
\end{aligned}$$

so that

$$S \lesssim \sum_{j \geq 0} \sum_{j'=j, j+1} \|f_{j'}\|_{H^\alpha} \|g_j\|_{H^\beta} \sum_{0 \leq l \leq j} 2^{-\delta l} \|h_l\|_{H^\gamma} \cdot 2^{-\alpha j' - \beta j - \gamma l + \frac{1}{2}(1+\varepsilon)l + \delta l}.$$

By applying the Cauchy-Schwarz inequality in j and l , it then suffices to show

$$p := -\alpha j' - \beta j - \gamma l + \frac{1}{2}(1+\varepsilon)l + \delta l \leq C$$

under the condition $0 \leq l \leq j \leq j' \leq j+1$. It is enough to consider the worst case that $\alpha \leq \beta \leq \gamma$.

If $\alpha \geq 0$, then $\alpha j' + \beta j + \gamma l \geq (\alpha + \beta + \gamma)l$, which implies $p \leq 0$.

If $\alpha < 0$, then by the assumption $\alpha + \beta \geq 0$, we have

$$\alpha j' + \beta j + \gamma l = \alpha(j' - j) + (\alpha + \beta)j + \gamma l \geq -|\alpha| + (\alpha + \beta + \gamma)l,$$

which implies $p \leq |\alpha|$.

This concludes the proof. \square

Remark 5.5. Lemma 5.1 also holds for any *rational* domains; $\mathbf{T}_a^3 = [0, 2\pi a_1) \times [0, 2\pi a_2) \times [0, 2\pi a_3)$ satisfying $a_2^2/a_1^2, a_3^2/a_1^2 \in \mathbb{Q}$. In fact, we may assume that $b_i := a_i^{-2} \in \mathbb{N}$, $i = 1, 2, 3$ by a scaling argument. Then, the resonance condition $\omega_{nkm}^\sigma = 0$ is replaced by

$$\sigma_1 \frac{k_3}{|\check{k}|} + \sigma_2 \frac{m_3}{|\check{m}|} = \sigma_3 \frac{n_3}{|\check{n}|}$$

with $|\check{k}|^2 := b_1 k_1^2 + b_2 k_2^2 + b_3 k_3^2$. Since $|\check{k}|^2 \in \mathbb{N}$ for any $k \in \mathbb{Z}^3$, most of the above argument is applicable, except that Lemma 5.3 (ii) should be modified as

$$\#\{(x, y) \in \mathbb{Z}^2 \mid b_1 x^2 + b_2 y^2 = N\} \leq C(\varepsilon, b_1, b_2) N^\varepsilon.$$

This is actually true for any $b_1, b_2 > 0$ by the result of Bombieri and Pila [4, Theorem 3]. Once we have the key estimate (Lemma 5.1), we can show the main result (Theorem 1.3) for rational domains by the same arguments with some trivial modifications.

6. ERROR ESTIMATE AND CONCLUSION

Let $u_0 \in H^1$ be arbitrarily large initial vector field which is real-valued, divergence-free and mean-zero, and let $E := \|u_0\|_{H^1}$. For $\alpha \in (\frac{3}{4}, 1]$, we fix an $\varepsilon \in (0, 2\alpha - \frac{3}{2})$. In this section, we consider the case

$$P := \frac{E}{\nu} \geq 1,$$

taking larger E if necessary. ($P \ll_\alpha 1$ corresponds to the small-data case, where we have a unique global solution to (1.1) for any $\Omega \in \mathbb{R}$.) The purpose of this section is to see how global smooth solutions of (1.5) (and hence, of (1.1)) are constructed from those of the limit equation (1.7) in the fast rotation case ($|\Omega| \geq \Omega_0 \gg 1$), and how Ω_0 depends on the initial vector field. In what follows, $C(\alpha)$ denotes any positive constant depending on α with $C(\alpha) \rightarrow \infty$ as $\alpha \downarrow \frac{3}{4}$ (hence $\varepsilon \downarrow 0$), while C denotes any absolute positive constant.

The main theorem (Theorem 1.3) follows once we have the same result for the equation (1.5), which will be shown by estimating the H^1 distance between solution $v(t)$ of (1.5) and the corresponding solution $U(t)$ of (1.7) up to some very large but finite time. We will need the uniform $H^{1+\gamma}$ bound on the solution $v(t)$ of (1.5) up to such a finite time, with $\gamma > 0$ satisfying $2\gamma + \alpha > \frac{3}{2}$. In fact, we will show that with $\gamma = \frac{1}{2}$.

By the proof of Theorem 1.2, we have the following: If the solution $v(t)$ of (1.5) belongs to H^s ($s \geq 1$) at an initial time t_0 , and if β satisfies the conditions (2.2), then for the local existence time given by

$$T_L = T_L(s, \alpha, \beta, \nu; \|v(t_0)\|_{H^s}) := c(s, \alpha, \beta) \nu^{-1} \left(\frac{\nu}{\|v(t_0)\|_{H^s}} \right)^{\frac{2\alpha}{2\alpha - (1+\beta)}},$$

it holds that

$$\|v(t_0 + t)\|_{H^{s+\beta}} \leq C(\nu t)^{-\frac{\beta}{2\alpha}} \|v(t_0)\|_{H^s}, \quad 0 < t \leq T_L.$$

Taking $s = 1$, $t_0 = 0$ and $\beta = \frac{1}{2}$, we have the local existence time in the first step:

$$T_L = T_L(1, \alpha, \frac{1}{2}, \nu; E) \leq C(\alpha)^{-1} E^{-1} P^{-C(\alpha)},$$

and the $H^{\frac{3}{2}}$ bound

$$\|v(T_L)\|_{H^{\frac{3}{2}}} \lesssim (\nu T_L)^{-\frac{1}{4\alpha}} E \leq C(\alpha) E P^{C(\alpha)},$$

as well as the H^1 bound

$$\|v(T_L)\|_{H^1} \leq C E.$$

Next, we consider the solution $U(t)$ of (1.7) with initial condition

$$U(T_L) = v(T_L).$$

By Proposition 1.7, $U(t)$ eventually becomes sufficiently small;

$$\|U(T_L + T_C)\|_{H^1} \leq \frac{1}{2} \eta(1, \alpha) \nu,$$

where $\eta(s, \alpha)$ is the small constant given in Theorem 1.2, and T_C is some large time satisfying

$$\begin{aligned} T_C &\leq C(\alpha, \varepsilon) \left(\frac{\eta(1, \alpha)\nu}{2} \right)^{-2} \frac{(CE)^2}{\nu} \left\{ 1 + \left(\frac{CE}{\nu} \right)^{\frac{4(1-\alpha)}{4\alpha-(3+2\varepsilon)}} \right\} \\ &\leq C(\alpha, \varepsilon) \nu^{-1} P^{2+\frac{4(1-\alpha)}{4\alpha-(3+2\varepsilon)}} = C(\alpha) \nu^{-1} P^{C(\alpha)}. \end{aligned}$$

We will see that $v(t)$ stays close to $U(t)$ until $t = T_L + T_C$ if $|\Omega|$ is sufficiently large. Precisely, we set

$$T_* := \sup \{ T \geq 0 \mid \|v(t) - U(t)\|_{H^1} \leq \frac{1}{2}\eta(1, \alpha)\nu, \quad t \in [T_L, T_L + T] \}$$

and show that $T_* \geq T_C$ if $|\Omega|$ is large enough. Since it will then hold that $\|v(T_L + T_C)\|_{H^1} \leq \eta(1, \alpha)\nu$, this will show the global existence of regular solutions of (1.5), hence those of (1.1).

By (4.7), we know that

$$(6.1) \quad \|U(t)\|_{H^1}^2 + \nu \int_{T_L}^t \|U(t')\|_{H^{1+\alpha}}^2 dt' \leq \tilde{E}^2 := C(\alpha, \varepsilon) E^2 P^{\frac{4}{4\alpha-(3+2\varepsilon)}}, \quad t \geq T_L.$$

Let \tilde{T}_L be the local existence time of the H^1 solution to (1.5) of size $2\tilde{E}$, namely,

$$\tilde{T}_L := T_L(1, \alpha, \frac{1}{2}, \nu; 2\tilde{E}) = C(\alpha)^{-1} \nu^{-1} \left(\frac{\nu}{2\tilde{E}} \right)^{\frac{4\alpha}{4\alpha-3}},$$

and let K be the smallest integer greater than T_C/\tilde{T}_L . We see that

$$(6.2) \quad K \leq C(\alpha, \varepsilon) P^{3+\frac{3}{4\alpha-3}+\frac{4}{4\alpha-(3+2\varepsilon)}} (1-\alpha+\frac{2\alpha}{4\alpha-3}) = C(\alpha) P^{C(\alpha)}.$$

By iterating the local bound (1.3), we obtain the a priori $H^{\frac{3}{2}}$ bound of $v(t)$: For $k = 0, 1, \dots, K-1$, if the H^1 bound

$$\|v(t)\|_{H^1} \leq 2\tilde{E}, \quad T_L \leq t \leq T_L + k\tilde{T}_L$$

holds true, then we have

$$\begin{aligned} (6.3) \quad &\|v(t)\|_{H^{\frac{3}{2}}} \leq C^{k+1} \|v(T_L)\|_{H^{\frac{3}{2}}} \leq C^K \cdot E \cdot C(\alpha) P^{C(\alpha)} \\ &\leq \exp((\log C)K) E \exp(C(\alpha) P^{C(\alpha)}) \leq E \exp(C(\alpha) P^{C(\alpha)}) \\ &=: L, \quad T_L \leq t \leq T_L + (k+1)\tilde{T}_L, \end{aligned}$$

where we have used (6.2) in the last inequality.

Now, we estimate the difference $w(t) := v(t) - U(t)$, which satisfies

$$(6.4) \quad \begin{cases} \partial_t w + \nu(-\Delta)^\alpha w + B_R(w, v) + B_R(U, w) + B_{NR}(\Omega t; v, v) = 0, & t > T_L, \\ w|_{t=T_L} = 0. \end{cases}$$

By the H^1 energy estimate, we have

$$\begin{aligned} \frac{d}{dt} \|w(t)\|_{H^1}^2 + 2\nu \|w(t)\|_{H^{1+\alpha}}^2 &\leq C \left(\|v(t)\|_{H^{\frac{3}{2}}} + \|U(t)\|_{H^{1+\alpha}} \right) \|w(t)\|_{H^1} \|w(t)\|_{H^{1+\alpha}} \\ &\quad - 2 \langle B_{NR}(\Omega t; v(t), v(t)), w(t) \rangle_{H^1}. \end{aligned}$$

Here, we have used the nonlinear estimate

$$\left| \langle B_R(f, g), h \rangle_{H^1} \right| \lesssim \|f\|_{H^{\frac{3}{2}}} \|g\|_{H^{\frac{3}{2}}} \|h\|_{H^1},$$

which is a consequence of the 2D-like estimate on the size of (trivial and non-trivial) resonant frequencies; see [3, Lemma 3.1]. Using Young's inequality, we have

$$(6.5) \quad \frac{d}{dt} \|w(t)\|_{H^1}^2 + \nu \|w(t)\|_{H^{1+\alpha}}^2 \leq C\nu^{-1} \left(\|v(t)\|_{H^{\frac{3}{2}}}^2 + \|U(t)\|_{H^{1+\alpha}}^2 \right) \|w(t)\|_{H^1}^2 - 2\langle B_{NR}(\Omega t; v(t), v(t)), w(t) \rangle_{H^1},$$

as long as the solution $v(t)$ (and thus $w(t)$) exists.

We will exploit the following lemma to control the non-resonant part:

Lemma 6.1. *For any $\delta > 0$, there exists $\Omega_0 = \Omega_0(\delta, \alpha, E, P) > 0$ such that the following holds for $|\Omega| \geq \Omega_0$. Let $k \in \{0, 1, \dots, K-1\}$. Assume that $\|v(t)\|_{H^1} \leq 2\tilde{E}$ for $T_L \leq t \leq T_L + k\tilde{T}_L$. Then, we have*

$$\begin{aligned} & \left| 2 \int_{T_L}^t \langle B_{NR}(\Omega t'; v(t'), v(t')), w(t') \rangle_{H^1} dt' \right| \\ & \leq \delta + \frac{1}{2} \left(\|w(t)\|_{H^1}^2 + \nu \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' \right) \end{aligned}$$

for $T_L \leq t \leq T_L + (k+1)\tilde{T}_L$. One can take Ω_0 as

$$\Omega_0 = E \left(\frac{E^2}{\delta} \right)^C \exp \left(C(\alpha) P^{C(\alpha)} \right).$$

Let us admit the above lemma and continue the proof. Assume that

$$(6.6) \quad \|v(t)\|_{H^1} \leq 2\tilde{E} \quad \text{for } T_L \leq t \leq T_L + k\tilde{T}_L, \quad T_* \geq k\tilde{T}_L$$

for some $k \in \{0, 1, \dots, K-1\}$. (This assumption is trivial for $k=0$.) Integrating (6.5) over $[T_L, t]$ and applying the lemma, as well as (6.3), we have

$$\begin{aligned} & \|w(t)\|_{H^1}^2 + \nu \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' \\ & \leq 2\delta + C\nu^{-1} \int_{T_L}^t \left(L^2 + \|U(t')\|_{H^{1+\alpha}}^2 \right) \|w(t')\|_{H^1}^2 dt' \end{aligned}$$

with $|\Omega| \geq \Omega_0$, up to $t = T_L + (k+1)\tilde{T}_L$. By the Gronwall inequality,

$$\|w(t)\|_{H^1}^2 + \nu \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' \leq 2\delta e^{C\nu^{-1}T_C L^2} \exp \left(C\nu^{-1} \int_{T_L}^t \|U(t')\|_{H^{1+\alpha}}^2 dt' \right).$$

We use the H^1 a priori estimate (6.1) on $U(t)$ to obtain

$$\begin{aligned} & \|w(t)\|_{H^1}^2 + \nu \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' \\ & \leq 2\delta \exp \left[C\nu^{-1}T_C L^2 + C \left(\frac{\tilde{E}}{\nu} \right)^2 \right] \leq \delta \exp \exp \left(C(\alpha) P^{C(\alpha)} \right). \end{aligned}$$

We finally take δ sufficiently small so that

$$\delta \exp \exp \left(C(\alpha) P^{C(\alpha)} \right) \leq \min \left\{ \frac{1}{2} \eta(1, \alpha) \nu, \tilde{E} \right\}^2 \sim C(\alpha)^{-1} \nu^2.$$

Then, (6.6) is also true for $k+1$, and for all k by induction. Ω_0 can be taken as

$$\begin{aligned} \Omega_0 &= E \left(C(\alpha) P^2 \exp \exp \left(C(\alpha) P^{C(\alpha)} \right) \right)^C \exp \left(C(\alpha) P^{C(\alpha)} \right) \\ &\leq E \exp \exp \left(C(\alpha) P^{C(\alpha)} \right). \end{aligned}$$

This concludes the proof of Theorem 1.3, up to the proof of Lemma 6.1.

Proof of Lemma 6.1. We will use the fundamental estimate

$$(6.7) \quad \begin{aligned} & \sum_{n,k \in \mathbb{Z}^3} |\widehat{f}(k)| |n-k| |\widehat{g}(n-k)| |n|^2 |\widehat{h}(n)| \\ & \leq \begin{cases} C(\alpha) \min \left\{ \|f\|_{H^{\frac{5}{4}}} \|g\|_{H^{\frac{3}{2}}} \|h\|_{H^{1+\alpha}}, \|f\|_{H^{\frac{3}{2}}} \|g\|_{H^{\frac{5}{4}}} \|h\|_{H^{1+\alpha}} \right\}, \\ C \|f\|_{H^{\frac{3}{2}}} \|g\|_{H^{\frac{3}{2}}} \|h\|_{H^{1+\alpha}}, \end{cases} \end{aligned}$$

which can be shown by the Hölder and the Young inequalities for $\alpha > \frac{3}{4}$.

First, we deduce from (6.7), (6.3) and the high-frequency estimate

$$\|P_{>N} v\|_{H^{\frac{5}{4}}} \leq N^{-\frac{1}{4}} \|\nabla^{\frac{1}{4}} v\|_{H^{\frac{5}{4}}} = N^{-\frac{1}{4}} \|v\|_{H^{\frac{3}{2}}},$$

that

$$\begin{aligned} & \left| \int_{T_L}^t \langle B_{NR}(\Omega t'; v(t'), v(t')) - B_{NR}(\Omega t'; P_{\leq N} v(t'), P_{\leq N} v(t')), w(t') \rangle_{H^1} dt' \right| \\ & \leq C(\alpha) N^{-\frac{1}{4}} L^2 \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}} dt' \\ & \leq \frac{\nu}{8} \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' + C(\alpha) \nu^{-1} T_C L^4 N^{-\frac{1}{2}} \end{aligned}$$

for $T_L \leq t \leq T_L + (k+1)\widetilde{T}_L$, where $P_{\leq N} = \mathcal{F}^{-1} \chi_{\{|n| \leq N\}} \mathcal{F}$. Therefore, the high-frequency part is manageable by taking

$$N = \delta^{-2} C(\alpha) \nu^{-2} T_C^2 L^8 \sim \delta^{-2} E^4 \exp \left(C(\alpha) P^{C(\alpha)} \right).$$

Let us fix such an N and proceed to the low-frequency part. Now, we claim that:

$$(6.8) \quad \inf \left\{ |\omega_{nk(n-k)}^\sigma| \mid \sigma \in \{\pm\}^3, n, k \in \mathbb{Z}^3 \setminus \{0\} \text{ s.t. } n \neq k, \omega_{nk(n-k)}^\sigma \neq 0, |k| \leq N, |n-k| \leq N \right\} \gtrsim N^{-12}.$$

Take $k, n \neq 0$ such that $|k|, |n-k| \leq N$ and $n-k \neq 0$. If there is a $\sigma \in \{\pm\}^3$ such that $\omega_{nk(n-k)}^\sigma = 0$, then we see that

$$\omega_{nk(n-k)}^\sigma \in \left\{ 0, \pm 2 \frac{k_3}{|k|}, \pm 2 \frac{n_3 - k_3}{|n-k|}, \pm 2 \frac{n_3}{|n|} \right\}$$

for any $\sigma \in \{\pm\}^3$. In this case, $|\omega_{nk(n-k)}^\sigma| \gtrsim N^{-1}$ unless $\omega_{nk(n-k)}^\sigma = 0$.

We thus assume that $\omega_{nk(n-k)}^\sigma \neq 0$ for all σ . In this case, by the identity

$$\begin{aligned} & \prod_{\sigma_1, \sigma_2 \in \{\pm\}} \omega_{nk(n-k)}^{\sigma_1, \sigma_2, -} \\ & = \frac{k_3^4}{|k|^4} + \frac{(n_3 - k_3)^4}{|n-k|^4} + \frac{n_3^4}{|n|^4} - 2 \left(\frac{k_3^2}{|k|^2} \frac{(n_3 - k_3)^2}{|n-k|^2} + \frac{n_3^2}{|n|^2} \frac{k_3^2}{|k|^2} + \frac{(n_3 - k_3)^2}{|n-k|^2} \frac{n_3^2}{|n|^2} \right) \\ & = \frac{(\text{non-zero integer})}{|k|^4 |n-k|^4 |n|^4} \end{aligned}$$

and the upper bound $|\omega_{nk(n-k)}^\sigma| \leq 3$, we have $|\omega_{nk(n-k)}^\sigma| \gtrsim N^{-12}$ for any σ . Therefore, (6.8) has been proved.

By integration by parts in t' , we see that

$$\begin{aligned}
& \int_{T_L}^t \langle B_{NR}(\Omega t'; P_{\leq N} v(t'), P_{\leq N} v(t')), w(t') \rangle_{H^1} dt' \\
&= i \int_{T_L}^t \sum_{\sigma} \sum_{\substack{n, k \in \mathbb{Z}^3 \setminus \{0\} \\ \omega_{nk(n-k)}^{\sigma} \neq 0 \\ |k|, |n-k| \leq N}} e^{-i\Omega t' \omega_{nk(n-k)}^{\sigma}} [\widehat{v}^{\sigma_1}(t', k) \cdot (n-k)] [\widehat{v}^{\sigma_2}(t', n-k) \cdot |n|^2 \widehat{w}^{\sigma_3}(t', n)^*] dt' \\
&= \left[\sum_{\sigma} \sum_{\substack{\omega_{nk(n-k)}^{\sigma} \neq 0 \\ |k|, |n-k| \leq N}} \frac{e^{-i\Omega t' \omega_{nk(n-k)}^{\sigma}}}{-\Omega \omega_{nk(n-k)}^{\sigma}} [\widehat{v}^{\sigma_1}(t', k) \cdot (n-k)] [\widehat{v}^{\sigma_2}(t', n-k) \cdot |n|^2 \widehat{w}^{\sigma_3}(t', n)^*] \right]_{T_L}^t \\
&\quad + \int_{T_L}^t \sum_{\sigma} \sum_{\substack{\omega_{nk(n-k)}^{\sigma} \neq 0 \\ |k|, |n-k| \leq N}} \frac{e^{-i\Omega t' \omega_{nk(n-k)}^{\sigma}}}{-\Omega \omega_{nk(n-k)}^{\sigma}} [\partial_{t'} \widehat{v}^{\sigma_1}(t', k) \cdot (n-k)] [\widehat{v}^{\sigma_2}(t', n-k) \cdot |n|^2 \widehat{w}^{\sigma_3}(t', n)^*] dt' \\
&\quad + \int_{T_L}^t \sum_{\sigma} \sum_{\substack{\omega_{nk(n-k)}^{\sigma} \neq 0 \\ |k|, |n-k| \leq N}} \frac{e^{-i\Omega t' \omega_{nk(n-k)}^{\sigma}}}{-\Omega \omega_{nk(n-k)}^{\sigma}} [\widehat{v}^{\sigma_1}(t', k) \cdot (n-k)] [\partial_{t'} \widehat{v}^{\sigma_2}(t', n-k) \cdot |n|^2 \widehat{w}^{\sigma_3}(t', n)^*] dt' \\
&\quad + \int_{T_L}^t \sum_{\sigma} \sum_{\substack{\omega_{nk(n-k)}^{\sigma} \neq 0 \\ |k|, |n-k| \leq N}} \frac{e^{-i\Omega t' \omega_{nk(n-k)}^{\sigma}}}{-\Omega \omega_{nk(n-k)}^{\sigma}} [\widehat{v}^{\sigma_1}(t', k) \cdot (n-k)] [\widehat{v}^{\sigma_2}(t', n-k) \cdot |n|^2 \partial_{t'} \widehat{w}^{\sigma_3}(t', n)^*] dt'.
\end{aligned}$$

We assume that $|\Omega|$ is greater than some Ω_0 to be determined. Invoking (6.8) and (6.7), we have

$$\begin{aligned}
& \left| \int_{T_L}^t \langle B_{NR}(\Omega t'; P_{\leq N} v(t'), P_{\leq N} v(t')), w(t') \rangle_{H^1} dt' \right| \\
&\leq \frac{CN^{12}}{\Omega_0} \left[\|v(t)\|_{H^{\frac{3}{2}}}^2 \|P_{\leq 2N} w(t)\|_{H^2} \right. \\
&\quad \left. + \int_{T_L}^t \left(\|P_{\leq N} \partial_{t'} v(t')\|_{H^{\frac{3}{2}}} \|v(t')\|_{H^{\frac{3}{2}}} \|w(t')\|_{H^{1+\alpha}} \right. \right. \\
&\quad \left. \left. + \|v(t')\|_{H^{\frac{3}{2}}}^2 \|P_{\leq 2N} \partial_{t'} w(t')\|_{H^2} \right) dt' \right].
\end{aligned}$$

Now, since all the functions are restricted onto low frequencies, time derivatives of v and w can be estimated by using the equations and Lemma 7.1, as follows:

$$\begin{aligned}
\|P_{\leq N} \partial_t v(t)\|_{H^{\frac{3}{2}}} &\leq N^2 \|\partial_t v(t)\|_{H^{-\frac{1}{2}}} = N^2 \|\nu(-\Delta)^{\alpha} v + B(\Omega t; v(t), v(t))\|_{H^{-\frac{1}{2}}} \\
&\leq N^2 \left(\nu \|v(t)\|_{H^{\frac{3}{2}}} + C \|v(t)\|_{H^{\frac{3}{2}}}^2 \right),
\end{aligned}$$

$$\begin{aligned}
\|P_{\leq 2N} \partial_t w(t)\|_{H^2} &\leq (2N)^2 \|\partial_t w(t)\|_{L^2} \\
&= (2N)^2 \|\nu(-\Delta)^{\alpha} w + B_R(w, v) + B_R(U, w) + B_{NR}(\Omega t; v, v)\|_{L^2} \\
&\leq (2N)^2 \left[\nu \|w\|_{H^{1+\alpha}} + C \left(\|v\|_{H^{\frac{3}{2}}} + \|U\|_{H^1} \right) \|w\|_{H^{1+\alpha}} + C \|v\|_{H^{\frac{3}{2}}}^2 \right].
\end{aligned}$$

By these estimates and Young's inequality, we obtain that

$$\begin{aligned}
& \left| \int_{T_L}^t \langle B_{NR}(\Omega t'; P_{\leq N} v(t'), P_{\leq N} v(t')), w(t') \rangle_{H^1} dt' \right| \\
& \leq \frac{1}{4} \|w(t)\|_{H^1}^2 + \frac{CN^{26}}{\Omega_0^2} L^4 + \frac{\nu}{16} \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' + \frac{CN^{28}}{\Omega_0^2} \nu^{-1} (\nu + L)^2 L^4 T_C \\
& \quad + \frac{\nu}{16} \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' + \frac{CN^{28}}{\Omega_0^2} \nu^{-1} (\nu + L + \tilde{E})^2 L^4 T_C + \frac{CN^{14}}{\Omega_0} L^4 T_C \\
& \leq \frac{1}{4} \|w(t)\|_{H^1}^2 + \frac{\nu}{8} \int_{T_L}^t \|w(t')\|_{H^{1+\alpha}}^2 dt' \\
& \quad + \left(\frac{\delta^{-28} E^{59}}{\Omega_0} + \frac{\delta^{-52} E^{108} + \delta^{-56} E^{116}}{\Omega_0^2} \right) \exp \left(C(\alpha) P^{C(\alpha)} \right)
\end{aligned}$$

for $T_L \leq t \leq T_L + (k+1)\tilde{T}_L$. Together with the high-frequency estimate, we obtain the desired result by taking

$$\Omega_0 \sim E \left(\frac{E^2}{\delta} \right)^{29} \exp \left(C(\alpha) P^{C(\alpha)} \right).$$

The proof of Lemma 6.1 is now complete. \square

7. APPENDIX

7.1. Sobolev estimate. Here, we recall the following Sobolev estimate.

Lemma 7.1. *The inequality*

$$|\langle fg, h \rangle_{L^2(\mathbb{T}^d)}| \leq C \|f\|_{H^{s_1}(\mathbb{T}^d)} \|g\|_{H^{s_2}(\mathbb{T}^d)} \|h\|_{H^{s_3}(\mathbb{T}^d)}$$

holds if and only if either of the following (a), (b) holds:

- (a) $s_1 + s_2 + s_3 \geq \frac{d}{2}$ and $s_1 + s_2 + s_3 > \max\{s_1, s_2, s_3\}$,
- (b) $s_1 + s_2 + s_3 > \frac{d}{2}$ and $s_1 + s_2 + s_3 \geq \max\{s_1, s_2, s_3\}$.

Proof. We shall prove only the ‘if’ part here. Since the H^s norm is invariant under the complex conjugate (or obviously if f, g, h are real-valued), we may assume that $s_1 \geq s_2 \geq s_3$. In this case, the condition “(a) or (b)” is reduced to “(a’) or (b’)” with

$$(a') \quad s_1 + s_2 + s_3 \geq \frac{d}{2} \text{ and } s_2 + s_3 > 0, \quad (b') \quad s_1 + s_2 + s_3 > \frac{d}{2} \text{ and } s_2 + s_3 = 0.$$

Case 1: $s_3 \geq 0$. If (a’) holds, then there exist $2 \leq p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{2}$ and $\frac{1}{2} - \frac{1}{p_j} \leq \frac{s_j}{d}$ ($j = 1, 2, 3$). Using the Hölder inequality and the Sobolev embedding, we have

$$|\langle fg, h \rangle_{L^2}| \leq \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p_3}} \leq C \|f\|_{H^{s_1}} \|g\|_{H^{s_2}} \|h\|_{H^{s_3}}.$$

If (b’) holds, then $s_2 = s_3 = 0$ and $s_1 > \frac{d}{2}$. In this case, we have

$$|\langle fg, h \rangle_{L^2}| \leq \|f\|_{L^\infty} \|g\|_{L^2} \|h\|_{L^2} \leq C \|f\|_{H^{s_1}} \|g\|_{H^{s_2}} \|h\|_{H^{s_3}}.$$

Case 2: $s_3 < 0$. We use Parseval's identity and the inequality

$$\langle \xi \rangle^{-s_3} \leq C (\langle \xi - \xi' \rangle^{-s_3} + \langle \xi' \rangle^{-s_3}), \quad \xi, \xi' \in \mathbb{R}^d$$

to obtain that

$$\begin{aligned}
|\langle fg, h \rangle_{L^2}| &= \left| \left\langle \sum_{k'} \widehat{f}(k') \widehat{g}(k - k'), \widehat{h}(k) \right\rangle_{\ell_k^2} \right| \\
&\leq C \left\langle \sum_{k'} (\langle k - k' \rangle^{-s_3} + \langle k' \rangle^{-s_3}) |\widehat{f}(k')| |\widehat{g}(k - k')|, \langle k \rangle^{s_3} |\widehat{h}(k)| \right\rangle_{\ell_k^2} \\
&\leq C \left(\langle f^\dagger \cdot \langle \nabla \rangle^{-s_3} g^\dagger, \langle \nabla \rangle^{s_3} h^\dagger \rangle_{L^2} + \langle \langle \nabla \rangle^{-s_3} f^\dagger \cdot g^\dagger, \langle \nabla \rangle^{s_3} h^\dagger \rangle_{L^2} \right),
\end{aligned}$$

where $f^\dagger = \mathcal{F}^{-1}[\widehat{f}]$. This case is therefore reduced to Case 1, since $s_1 + s_3 \geq s_2 + s_3 \geq 0$ by the assumption, and the H^s norm is also invariant under the \dagger operation. \square

7.2. Scaling invariance and optimality of the result. We first recall the scaling invariance of the fractional Navier-Stokes equations. If (u, p) is a solution of (1.1) with $\Omega = 0$, then (u_λ, p_λ) with

$$u_\lambda(t, x) := \lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x), \quad p_\lambda(t, x) := \lambda^{4\alpha-2} p(\lambda^{2\alpha} t, \lambda x), \quad \lambda > 0$$

is also a solution with rescaled initial data

$$u_{0,\lambda}(x) := \lambda^{2\alpha-1} u_0(\lambda x)$$

with $\operatorname{div} u_{0,\lambda} = 0$. Although such a rescaling changes the period of spatial domain, one can still consider the scaling critical regularity s_c for which $\|u_{0,\lambda}\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$ for any $\lambda > 0$. We find that

$$s_c = \frac{d}{2} + 1 - 2\alpha.$$

By the scaling heuristics, the local-in-time theory may be developed in H^s for $s \geq s_c$ (sub-critical and critical regularities); equivalently, in H^s with a fixed regularity s for

$$(7.1) \quad \alpha \geq \frac{d + 2 - 2s}{4}.$$

For instance, local theory in H^1 requires $\alpha \geq \frac{1}{2}$ in 2D and $\alpha \geq \frac{3}{4}$ in 3D, as observed in the proof of Theorem 1.2.

Another regularity restriction may arise in the global a priori estimate for the limit equation (1.7). For the 2D part $\overline{U}(t)$, one can apparently gain one spatial derivative through the vorticity formulation; in fact, the nonlinear term $(\overline{U}^h \cdot \nabla^h) \omega$ has the same scaling as square of ω with no derivative. Then, the 2D part may have an H^s global a priori bound if the dissipation $\langle (-\Delta)^\alpha \omega, \omega \rangle_{H^s}$ dominates the nonlinearity $\langle \omega \omega, \omega \rangle_{H^s}$ in the energy estimate. Regarding an extra w in the nonlinearity as extra $\frac{d}{2}$ derivatives (by the scaling heuristics $L^\infty \sim H^{\frac{d}{2}}$) and compare the total number of derivatives in these terms, we find the condition

$$2\alpha + 2s > 2s + \frac{d}{2}.$$

Since \overline{U} behaves as a 2D flow, we set $d = 2$ to come to the condition $\alpha > \frac{1}{2}$.

For the oscillating part U_{osc} , the nonlinearity, which is quadratic with one derivative, has $1 + \varepsilon$ dimensional interactions, as suggested in Lemma 5.1. We compare the number of derivatives just as above, but with $d = 1 + \varepsilon$, to see that the condition

$$(7.2) \quad 2\alpha + 2s > 2s + 1 + \frac{1 + \varepsilon}{2} \quad \Leftrightarrow \quad \alpha > \frac{3}{4}$$

is required for an H^s global control on U_{osc} .

We remark that our result (Theorem 1.3), global regularity in H^1 for $\alpha > \frac{3}{4}$, is optimal in both (7.1) and (7.2). In other words, by (7.1) one needs regularity H^1 to deal with α arbitrarily close to $\frac{3}{4}$; however, one may not relax the condition on α due to the restriction (7.2) even if the initial data is more regular than H^1 . That is exactly why we work in H^1 in this article.

Hence, if we could prove Lemma 5.1 with just CL^ε in the right-hand side, then the restriction on α would be relaxed to $\alpha > \frac{1}{2}$. In this case, however, one has to work with higher regularity $H^{\frac{3}{2}}$, due to (7.1). Another natural space to work in is the Fourier-Lebesgue space $\mathcal{F}^{-1}\ell^1(\mathbb{T}^3)$ defined in Remark 1.6. This space has the same scaling as $H^{\frac{3}{2}}(\mathbb{T}^3)$, while it is an algebra and continuously embedded into the space of (bounded uniformly) continuous functions on \mathbb{T}^3 .

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